Nonlinear Analysis 73 (2010) 31–47

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A short proof of the $C^{0,\alpha}$-regularity of viscosity subsolutions for superquadratic viscous Hamilton–Jacobi equations and applications[☆]

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ARTICLE INFO

Article history:
Received 24 April 2009
Accepted 10 February 2010

MSC:
35J60
35B65
35J65
35J70
49L25

Keywords:
Hölder regularity
Fully nonlinear equations
Ergodic problems
Homogenization
Viscosity solutions

ABSTRACT

Recently I. Capuzzo Dolcetta, F. Leoni and A. Porretta obtained a very surprising regularity result for fully nonlinear, superquadratic, elliptic equations by showing that viscosity subsolutions of such equations are locally Hölder continuous, and even globally if the boundary of the domain is regular enough. The aim of this paper is to provide a simplified proof of their results, together with an interpretation of the regularity phenomena, some extensions and various applications.

1. Introduction

In [1], Capuzzo Dolcetta, Leoni and Porretta obtain a very surprising regularity result for fully nonlinear, *superquadratic*, elliptic equations which is described very easily in their main example which is the one of viscous Hamilton–Jacobi Equations

$$-\operatorname{Tr}(A(x)D^2u) + |Du|^p + \lambda u = f(x) \quad \text{in } \Omega, \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^N$, $\lambda \geq 0$, $p > 2$ and A, f are continuous functions taking values respectively in the set of non-negative, $N \times N$ symmetric matrices and $\mathbb{R}$. They show that if u is a locally bounded, upper semicontinuous viscosity subsolution of (1.1) then u is locally Hölder continuous with exponent $\alpha = \frac{p-2}{p-1}$. Furthermore they prove that the local $C^{0,\alpha}$-bound depends only on local L^∞-bounds on A, f and λu^-. They also provide global regularity results in the case when the boundary $\partial\Omega$ has a sufficient regularity, namely when the boundary is Lipschitz continuous and satisfies an interior sphere condition.

These results are very unusual and surprising since they provide the regularity of *subsolutions* of *degenerate* equations with a *superquadratic* growth in Du, whereas most of the regularity results for elliptic equations concern *solutions* of *uniformly elliptic* equations with suitable (subquadratic) *growth conditions*. At this point, it is worth mentioning the famous work of Lasry and Lions [2] where Eq. (1.1) is studied in full detail, both in the sub and superquadratic cases, when the second-order

☆ This work was partially supported by the ANR "Hamilton-Jacobi et théorie KAM faible" (ANR-07-BLAN-3-187245).
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doi:10.1016/j.na.2010.02.009

term is the laplacian ($A \equiv Id$): several local gradient bounds are provided by using Bernstein's method (see also [3] for results in this directions) together with various estimates on the solutions, and all these properties are used to prove existence, uniqueness results in different contexts (infinite boundary conditions, data which are blowing up at the boundary, ergodic problem, …). Most probably, some of their results are still true for (1.1) even if we allow A to degenerate but, at least, their regularity results are valid only for solutions.

Coming back to [1], the way the authors explain it is through the case $A \equiv 0$ for which one has obviously a Lipschitz bound for subsolutions (since Du is clearly bounded) and for general A, the viscous Hamilton–Jacobi Equation can be seen as a perturbation of the first-order equation and keeps, at least partially, a similar property when the power of Du is large enough, namely larger than 2.

The aim of the present paper is threefold:

(i) to give a slightly simpler proof of this result in a more general setting,
(ii) to provide an interpretation of such property in terms of "state-constraint problems",
(iii) to use this result to obtain, for superquadratic equations, new results for the generalized Dirichlet problem (in the sense of viscosity solutions), for ergodic problems and homogenization problems.

In order to be more specific, we come back to Eq. (1.1) and we examine again the case $A \equiv 0$: if u is a subsolution of this equation, then

$$|Du|^p \leq f(x) - \lambda u \quad \text{in } \Omega,$$

and if we assume also that $\lambda = 0$, we have a gradient bound which is independent of the L^∞-norm of u. And the same property is true for $\lambda \neq 0$ if u is bounded from below.

One does not expect such property to be true for elliptic equations and, in general, all the $C^{0,\alpha}$ or Lipschitz bounds depend on (local) L^∞-bounds on u. But, as we already mention it above, the authors prove in [1] that the $C^{0,\alpha}$-bound is still true for general A under the same conditions as for the first-order equation.

Our approach, whose general framework is described in Section 2.1, shows why both situations are very similar: roughly speaking, if u is a subsolution of a general equation, we are not going to argue directly on this equation but on a simpler equation for which u is still a subsolution; for the above first-order equation, clearly the only important information is that

$$|Du|^p \leq \|f\|_\infty + \|\lambda u^-\|_\infty,$$

where the L^∞-norm is either a local or a global norm, and this step can be seen as a replacement of a complicated equation by a simpler one. As this (very simple) example shows it, this replacement may depend on (local) L^∞-bounds of u (in the case $\lambda \neq 0$) but once this step is done then the (local) L^∞-bounds will play not role anymore.

In order to obtain the $C^{0,\alpha}$-bounds, the key argument consists in building, for the new equation, a family of supersolutions $(w_r)_r$ in balls of radius $r \ll 1$: these functions are used to control from above the local variations of the subsolution and, of course, this control gives the Hölder regularity. Two points have to be emphasized: first, the w_r are constructed in such a way that they are supersolutions up to the boundary of the balls (this is called "state-constraints boundary conditions") and this point is crucial to have a control of the subsolution which is independent of its L^∞-bounds when the new equation does not depend on such L^∞-bounds (the case when $\lambda = 0$ in (1.1)). Next the construction of such a family of w_r is possible only in the superquadratic case: we address this question, with several variants, at the end of Section 2.1.

Therefore, in "good" cases (typically when $\lambda = 0$ in (1.1)), one can obtain $C^{0,\alpha}$-bounds which are independent of any L^∞-bounds on the subsolution and if $\partial\Omega$ is regular enough, these bounds hold up to the boundary. Section 2.2 is devoted to provide various examples of equations to which the framework of Section 2.1 applies and we formulate a general result in Section 2.3 in which we obtain local modulus of continuity which are not necessarily of Hölder type.

Concerning the applications, we are not going to describe them in this introduction; we refer the reader to the corresponding sections. Section 3 is devoted to the study of the generalized Dirichlet problem for general superquadratic elliptic equations: assuming or not that the equation is uniformly elliptic, one cannot solve in general the classical Dirichlet problem: we refer for example to Da Lio and the author [4] where the evolution problem is studied and where it is shown that loss of boundary data can occur. For (1.1), it is even obvious that the Dirichlet problem cannot be solved in a classical way since, for smooth enough boundary, the solution is expected to be $C^{0,\alpha}$ up to the boundary and therefore a solution of the classical Dirichlet problem can exist only in cases when the boundary data satisfies rather restrictive conditions. We refer to [1] where this question in studied in full details. On the contrary, we concentrate on solving the generalized Dirichlet problem in the sense of viscosity solutions. The role of $C^{0,\alpha}$-property in this setting is to provide the continuity up to the boundary of the subsolutions which is a key property to obtain comparison results. We refer to [5–8] for more details.

For ergodic and homogenization problems, the role of $C^{0,\alpha}$-bounds is well-known: it is a key argument to solve ergodic problem/cell problem and we show how this can be done for superquadratic equations in Sections 4 and 5. We refer to the bibliography for various references on ergodic and homogenization problems.