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The inverse symmetry problem for a 2*D* generalized second order evolutionary equation

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ABSTRACT

The paper is devoted to the Lie symmetries of 2*D* nonlinear dynamical systems described by second order partial differential equations. By imposing the invariance condition of the equations under the action of the Lie symmetry operator we obtained a determining system which could be solved in two directions: (i) to find the symmetries of a concrete equation; (ii) to obtain all the compatible equations with an imposed form of symmetry algebra. This paper will pay attention to the indirect problem (ii) and will use the algorithm for determining the most general 2*D* equations which belong, following their symmetries, to the same class with two interesting and important physical models: the nonlinear heat equation and the transfer equation with power-law nonlinearities.

1. Introduction

The Lie theory of symmetry groups for differential equations has been proven to be a powerful tool for studying nonlinear problems which appear in physics or in other fields of applied mathematics [1–3]. The main idea of Lie's theory is to investigate integrability starting from the invariance of the equation under some linear transformations of the variables, transformations which define the so-called Lie group of symmetries. There are various types of symmetries which can be identified for various differential equations: point symmetries, contact symmetries, classical, generalized or non-local symmetries [4–6].

Usually, *the direct symmetry problem* of evolutionary equations is considered. It consists in determining the symmetries of a given evolutionary equation and, by using them, to find attached invariants or conservation laws. The specific symmetries of each differential equation provide important information on the integrability of the equation, or help in finding some of its particular solutions. This paper will concentrate mainly on the *inverse symmetry problem*. We raise the question what is the largest class of evolutionary equations which are equivalent from the point of view of their symmetries. The same symmetries mean similar invariants and, by similarity reduction, similarly reduced equations. So, the inverse symmetry problem allows in fact to split a general class of evolutionary equations into many equivalent classes of equations as far as their invariants and reduced equations are concerned. It is interesting to remark that the well-known equations, apparently with no connection among themselves, belong to the same class from this point of view.

We shall illustrate this assertion by considering a general class of second order partial differential equations defined in a $(2+1)$-dimensional space–time, with the independent variables x, y, t and with only one dependent variable, $u = u(x, y, t)$. More precisely, we shall consider an evolutionary equation of the generic form $u_t = F(t, x, y, u, u_x, u_y, u_{2x}, u_{2y}, u_{xy})$. We shall propose an algorithm which will allow us to identify the most general form of these types of equations which admit

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the same given symmetry group. We will choose a symmetry group corresponding to a linear sector, group which generates an 8D Lie algebra. The general problem will be formulated in the next section of the paper and, in Section 3, the case of a Lie symmetry operator with linear coefficient functions will be analyzed. The results will be specified in Section 4 of the paper for two 2*D* models represented by the *nonlinear heat equation and (mass/heat) transfer equation with power-law nonlinearities*. These nonlinear equations correspond to Lie subalgebras of smaller dimensions. They have always played an important role in the formation of a correct understanding of qualitative features of various transport processes in chemical engineering, thermophysics, and power engineering [7]. In non-homogeneous media, the diffusion coefficients may depend on coordinates and even on temperature. There are numerous approximation formulas (among them linear, power-law and exponential) describing the dependence of the transfer coefficients on temperature or concentration [8]. On the other hand, we shall see that a model arising from a completely different field, the theory of gravity, namely the Ricci flow model [9] belongs, as a particular case, to the same class of equations. For each equation we shall consider, the form of the solutions will be investigated. Some concluding remarks will end the paper.

2. The general setting of the symmetry problem

Let us consider a 2*D* dynamical system described by a second order partial differential equation of the general form:

$$\begin{aligned} u_t &= A(x,y,t,u)u_{xy} + B(x,y,t,u)u_x u_y + C(x,y,t,u)u_{2x} + D(x,y,t,u)u_{2y} \\ &\quad + E(x,y,t,u)u_y + F(x,y,t,u)u_x + G(x,y,t,u) \end{aligned} \tag{1}$$

with $A(x,y,t,u)$, $B(x,y,t,u)$, $C(x,y,t,u)$, $D(x,y,t,u)$, $E(x,y,t,u)$, $F(x,y,t,u)$, $G(x,y,t,u)$ arbitrary functions of their arguments.

The general expression of the classical Lie operator which leaves (1) invariant is:

$$U(x,y,t,u) = \varphi(x,y,t,u)\frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u}. \tag{2}$$

Following the Lie symmetry theory [10], the invariance condition of the Eq. (1) is given by the relation:

$$\begin{aligned} 0 &= U^{(2)}[u_t - A(x,y,t,u)u_{xy} - B(x,y,t,u)u_x u_y - C(x,y,t,u)u_{2x} - D(x,y,t,u)u_{2y} \\ &\quad - E(x,y,t,u)u_y - F(x,y,t,u)u_x - G(x,y,t,u)] \end{aligned} \tag{3}$$

where $U^{(2)}$ is the second order prolongation of the operator (2).

The latter relation has the equivalent expression:

$$\begin{aligned} 0 &= -A_t\varphi u_{xy} - B_t\varphi u_x u_y - C_t\varphi u_{2x} - D_t\varphi u_{2y} - E_t\varphi u_y - F_t\varphi u_x - G_t\varphi - A_x\xi u_{xy} - B_x\xi u_x u_y \\ &\quad - C_x\xi u_{2x} - D_x\xi u_{2y} - E_x\xi u_y - F_x\xi u_x - G_x\xi - A_y\eta u_{xy} - B_y\eta u_x u_y - C_y\eta u_{2x} - D_y\eta u_{2y} \\ &\quad - E_y\eta u_y - F_y\eta u_x - G_y\eta - A_u\phi u_{xy} - B_u\phi u_x u_y - C_u\phi u_{2x} - D_u\phi u_{2y} - E_u\phi u_y - F_u\phi u_x \\ &\quad - G_u\phi + \phi^t - A\phi^{xy} - C\phi^{2x} - D\phi^{2y} - B\phi^x u_y - F\phi^x - B\phi^y u_x - E\phi^y. \end{aligned} \tag{4}$$

The functions ϕ^t, ϕ^x, ϕ^y, ϕ^{2x}, ϕ^{2y}, ϕ^{xy} will be determined using the general formulas:

$$\begin{aligned} \phi^t &= \mathcal{D}_t[\phi - \varphi u_t - \xi u_x - \eta u_y] + \varphi u_{2t} + \xi u_{xt} + \eta u_{yt} \\ \phi^x &= \mathcal{D}_x[\phi - \varphi u_t - \xi u_x - \eta u_y] + \varphi u_{tx} + \xi u_{2x} + \eta u_{xy} \\ \phi^y &= \mathcal{D}_y[\phi - \varphi u_t - \xi u_x - \eta u_y] + \varphi u_{ty} + \xi u_{xy} + \eta u_{2y} \\ \phi^{xy} &= \mathcal{D}_{xy}[\phi - \varphi u_t - \xi u_x - \eta u_y] + \varphi u_{txy} + \xi u_{xxy} + \eta u_{xyy} \\ \phi^{2x} &= \mathcal{D}_{2x}[\phi - \varphi u_t - \xi u_x - \eta u_y] + \varphi u_{txx} + \xi u_{xxx} + \eta u_{xxy} \\ \phi^{2y} &= \mathcal{D}_{2y}[\phi - \varphi u_t - \xi u_x - \eta u_y] + \varphi u_{tyy} + \xi u_{xyy} + \eta u_{yyy}. \end{aligned} \tag{5}$$

By extending the relations (5), substituting them into the condition (4) and then equating to zero the coefficient functions of various monomials in derivatives of u, the following partial differential system with 24 equations is obtained:

$$\begin{aligned} &[A^2 + 2CD]\varphi_x + 3AD\varphi_y = 0 \\ &[A^2 + 2DC]\varphi_y + 3AC\varphi_x = 0 \\ &A\varphi_x + 2D\varphi_y = 0 \\ &\varphi_u = 0 \\ &\xi_{2u} = 0 \\ &\eta_{2u} = 0 \\ &A\varphi_y + 2C\varphi_x = 0 \end{aligned}$$