Nonlinear Analysis 73 (2010) 221–227

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A characterization of the minimal invariant sets of Alspach's mapping

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ARTICLE INFO

Article history:
Received 4 November 2009
Accepted 18 March 2010

MSC:
primary 46E30

Keywords:
Alspach's mapping
Nonexpansive mapping
Fixed point free
Weakly compact
Convex set
Minimal invariant sets
Baker's transform
Strongly mixing

ABSTRACT

In 1981, Dale Alspach modified the baker's transform to produce the first example of a nonexpansive mapping T on a weakly compact convex subset C of a Banach space that is fixed point free. By Zorn's lemma, there exist minimal weakly compact, convex subsets of C which are invariant under T and are fixed point free.

In this paper we produce an explicit formula for the nth power of T, T^n, and prove that the sequence $(T^n f)_{n\in\mathbb{N}}$ converges weakly to $\|f\|_1\,\chi_{[0,1]}$, for all $f\in C$. From this we derive a characterization of the minimal invariant sets of T.

1. Introduction

In 1981, Dale Alspach modified the baker's transform (or baker's transformation) from ergodic theory, to produce an example of a nonexpansive mapping on a weakly compact, convex subset of $L^1[0,1]$ that is fixed point free [1]. There are several examples of non-expansive mappings on weakly compact, convex sets that are fixed point free [2–4]. Interestingly, each of these mappings involves or resembles Alspach's example. So, Alspach's example remains the typical example of such a pair consisting of a mapping and a set.

Let $(X, \|\cdot\|)$ be a Banach space and $B\subseteq X$. Recall that $U : B\to B$ is said to be *nonexpansive* if

$$\|U(x)-U(y)\|\le\|x-y\|,\quad \text{for all } x,y\in B.$$

We assume that B is nonempty, bounded, closed and convex. A set $D\subseteq B$ is said to be U-*invariant* if $U(D)\subseteq D$. Nonempty, closed, convex, U-invariant subsets of B are of interest. In particular, a nonempty, closed, convex, U-invariant set $D\subseteq B$ is said to be *minimal invariant* if whenever $A\subseteq D$ is nonempty, closed, convex and U-invariant, it follows that $A=D$. Minimal invariant sets are in this sense the smallest U-invariant subsets of B. Clearly, the singleton containing any fixed point of U is minimal invariant. In this way, minimal invariant sets generalize the concept of fixed points. For more on minimal invariant sets of nonexpansive mappings we refer the reader to [5,6].

For any nonexpansive fixed point free mapping on a weakly compact, convex set, there exist a minimal invariant subset of positive diameter, by an application of Zorn's lemma [7]. These minimal invariant sets have not previously been explicitly characterized for Alspach's example or any other such mapping [6,5,8]. We will describe all minimal invariant sets of Alspach's mapping, T. The general idea will be to find a formula for T^n. Next, we will show that $(T^n f)_{n\in\mathbb{N}}$ converges weakly for all $f\in C$. Then, we will use [6] to provide a description of all the minimal invariant sets of T.

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doi:10.1016/j.na.2010.03.015

We note that the results in this paper, except for Section 4, form part of the first author's Ph.D. Thesis [9].

2. Preliminaries

We will denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}$ the set of all integers. As usual, $\mathbb{R}$ is the set of all real numbers. We begin with some definitions. For all $n \in \mathbb{N}$, for all $i \in \Delta_n := \{0, \ldots, 2^n - 1\}$, define $E_{(i,n)} := \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$. Also, let

$$C := \left\{f \in L^1[0,1] : 0 \le f(x) \le 1, \forall x \in [0,1]\right\}$$

and

$$S := \left\{s \in L^1[0,1] : s = \sum_{i=1}^{n} a_i \chi_{E_{(i,n)}} \text{ where } a_i \in \mathbb{R} \text{ and } n \in \mathbb{N}\right\}.$$

Next, for all $\alpha, \beta \in \mathbb{R}$, $\alpha \wedge \beta := \min\{\alpha, \beta\}$ and $\alpha \vee \beta := \max\{\alpha, \beta\}$. Fix $a, b \in \mathbb{R}$ with $a < b$ and fix $c \in \mathbb{R}$. Then we have the modular property:

$$(a \vee c) \wedge b = a \vee (c \wedge b).$$

Define $\mathrm{cut}_0(a, b, c) := (a \vee c) \wedge b$. Note that $\mathrm{cut}_0(a, b, c) = a$, if $c < a$; $\mathrm{cut}_0(a, b, c) = c$, if $a \le c \le b$; and $\mathrm{cut}_0(a, b, c) = b$, if $c > b$. We further define

$$\mathrm{cut}(a, b, c) := \mathrm{cut}_0(a, b, c) - a = ((a \vee c) \wedge b) - a = (a \vee (c \wedge b)) - a.$$

The following lemma contains a few properties of the cut function that we will use later.

Lemma 2.1. *Fix $a, b \in \mathbb{R}$ with $a < b$ and fix $c \in \mathbb{R}$.*
(1) *Fix arbitrary real-valued, Lebesgue-measurable functions f and g on $[0, 1]$. Let $E := \mathrm{supp}(f)$ and $F := \mathrm{supp}(g)$ and suppose that $E \cap F$ has Lebesgue measure zero. Then, for all $x \in [0, 1]$,*

$$\mathrm{cut}(a, b, f(x) + g(x)) = \mathrm{cut}(a, b, f(x))\chi_E(x) + \mathrm{cut}(a, b, g(x))\chi_F(x).$$

(2) *For all $t > 0$,*

$$\mathrm{cut}(ta, tb, tc) = t\,\mathrm{cut}(a, b, c).$$

(3) *For all $p, q \in \mathbb{R}$ with $0 \le p < q \le b - a$,*

$$\mathrm{cut}(p, q, \mathrm{cut}(a, b, c)) = \mathrm{cut}(a + p, a + q, c).$$

Proof. Properties (1) and (2) are easy to check. Let us see why (3) holds.
Fix $p, q \in \mathbb{R}$ with $0 \le p < q \le b - a$. Then

$$\begin{aligned}
\mathrm{cut}(p, q, \mathrm{cut}(a, b, c)) &= (p \vee \mathrm{cut}(a, b, c)) \wedge q - p \\
&= (p \vee [a \vee (c \wedge b) - a]) \wedge q - p \\
&= ((p + a) \vee [a \vee (c \wedge b)] - a) \wedge q - p \\
&= ((p + a) \vee [a \vee (c \wedge b)]) \wedge (q + a) - a - p \\
&= ((a + p) \vee a \vee (c \wedge b)) \wedge (a + q) - (a + p).
\end{aligned}$$

Recall that $p \ge 0$, and so $a + p \ge a$. Therefore,

$$\mathrm{cut}(p, q, \mathrm{cut}(a, b, c)) = ((a + p) \vee (c \wedge b)) \wedge (a + q) - (a + p).$$

But $p < b - a \iff a + p < b$. By the modular property, and the fact that $a + q \le b$,

$$\begin{aligned}
\mathrm{cut}(p, q, \mathrm{cut}(a, b, c)) &= (((a + p) \vee c) \wedge b) \wedge (a + q) - (a + p) \\
&= ((a + p) \vee c) \wedge b \wedge (a + q) - (a + p) \\
&= ((a + p) \vee c) \wedge (a + q) - (a + p) \\
&= \mathrm{cut}(a + p, a + q, c). \quad \square
\end{aligned}$$

Throughout this paper, we will extend real-valued, measurable functions f on $[0, 1]$ to $\mathbb{R}$ by defining $f(x) := 0$ for $x \in \mathbb{R} \setminus [0, 1]$. We define the mapping $T : C \to C$ in the following way. For all $f \in C$, for each $x \in [0, 1]$,

$$\begin{aligned}
(Tf)(x) &:= \mathrm{cut}\,(0, 1, 2f\,(2x))\,\chi_{E_{(0,1)}}(x) + \mathrm{cut}\,(1, 2, 2f\,(2x - 1))\,\chi_{E_{(1,1)}}(x) \\
&= ((0 \vee 2f(2x)) \wedge 1 - 0)\,\chi_{[0,1/2)}(x) + ((1 \vee 2f(2x - 1)) \wedge 2 - 1)\,\chi_{[1/2,1)}(x) \\
&= (2f(2x) \wedge 1)\,\chi_{[0,1/2)}(x) + ((2f(2x - 1) \vee 1) - 1)\,\chi_{[1/2,1)}(x).
\end{aligned}$$