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Non-isochronicity of the center in polynomial Hamiltonian systems[☆]

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ABSTRACT

In 2002 Jarque and Villadelprat proved that planar polynomial Hamiltonian systems of degree 4 have no isochronous centers and raised an open question for general planar polynomial Hamiltonian systems of even degree. Recently, it was proved that a planar polynomial Hamiltonian system is non-isochronous if a quantity, denoted by M_{2m-2}, can be computed such that $M_{2m-2} \leq 0$. As a corollary of this criterion, the open question was answered for those systems with only even degree nonlinearities. In this paper we consider the case of $M_{2m-2} > 0$ and give a new criterion for non-isochronicity. Applying the new criterion, we also answer the open question for some cases in which some terms of odd degree are included.

1. Introduction

Identifying isochronous centers in systems with a center is an interesting problem in the research into planar vector fields. Since the classification of quadratic systems for isochronous centers was completed by Loud [1] in 1964, efforts were also made for higher degree polynomial systems. For systems with homogeneous nonlinearities, Pleshkan [2] gave a complete classification of isochronous centers in the cubic case in 1969. Later Chavarriga et al. considered this problem for fourth-degree systems and fifth-degree ones in [3] and [4] respectively. Recently, the cases of fourth degree and fifth degree were also discussed in [5,6]. For systems with nonhomogeneous nonlinearities, some classifications of isochronous reversible cubic systems were given in [7,8]. The cubic Kukles isochronous centers were classified completely by Christopher and Devlin in [9]. Cubic polynomial Hamiltonian isochronous centers were classified by Cima et al. in [10]. Liénard isochronous centers with odd nonlinearities were classified by Sabatini [11], while a complete classification is still absent. On the other hand, some efforts were also made to answer the question of non-isochronicity for the polynomial Hamiltonian system

$$\dot{x} = -H_y(x, y), \qquad \dot{y} = H_x(x, y). \tag{1.1}$$

For system (1.1) with homogeneous nonlinearities, Schuman [12], Christopher and Devlin [9] and Gasull et al. [13] used Birkhoff's normal form, geometrical and dynamical methods and a method of computation for period coefficients respectively to prove that the origin cannot be an isochronous center. In 2000, Cima et al. [14] showed that system (1.1) in which $H(x, y)$ is of the special form $H(x, y) = F(x) + G(y)$ is not isochronous, except that system (1.1) is a linear one. In 2002 Jarque and Villadelprat proved that every center of a planar polynomial Hamiltonian differential system of degree 4 is non-isochronous [15]. Moreover, they raised a question: *Does there exist a planar polynomial Hamiltonian system of even*

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degree having an isochronous center? In 2007 Chen et al. [16] considered the Hamiltonian system (1.1) with the Hamiltonian

$$H(x,y) = \frac{x^2+y^2}{2} + \sum_{\substack{i+j=m \\ i,j\ge 0}}^{n} a_{i,j}x^i y^j, \qquad \sum_{\substack{i+j=m \\ i,j\ge 0}} a_{i,j}^2 \neq 0, \tag{1.2}$$

where $x, y, a_{i,j} \in \mathbf{R}$, $m \in \mathbb{N}$ (the set of all positive integers) with $n \geq m \geq 3$. They gave a quantity $M_{2k} := \sum_{i=0}^{k} a_{2k-2i,2i}\varpi_{i,k-i}$, where

$$\varpi_{i,j} := \frac{(2i)!(2j)!\pi}{i!j!(i+j)!2^{2i+2j+1}} \tag{1.3}$$

and $a_{2k-2i,2i} = 0$ for $2k < m$ or $2k > n$, and proved that the planar polynomial Hamiltonian system is non-isochronous if $M_{2m-2} \leq 0$. As a corollary of this criterion, Jarque and Villadelprat's question was answered for those systems with *only even degree* nonlinearities.

In this paper, we consider the same planar polynomial Hamiltonian system (1.1) with (1.2) in the case opposite to that of [16], i.e., $M_{2m-2} > 0$. As shown in [16–18], the main task remains to compute and discuss period coefficients. In our case those techniques of symbolic computation are not available for finding a set of generators of ideals of period coefficients with simpler expressions because the degrees m and n of polynomials are not specified. Moreover, it suffices to discuss period coefficients of order $\leq 2m-4$ in [16] but in our case we need to compute many more period coefficients of higher order, to which the method of computation used in [16] is hardly applicable. In this paper the conservativeness of the Hamiltonian system is utilized for finding those generators of the ideals with simpler expressions. We obtain expressions for those generators and give a necessary and sufficient condition for non-isochronicity of the system, which actually generalizes the main theorem of [16]. As a corollary, a sufficient condition is given for the center O to be non-isochronous; the condition cannot be dealt with in [16] but can be verified easily. We also use our new criterion to prove the non-isochronicity for some polynomial Hamiltonian systems of even degree in which some terms of odd degree are included. This case is concerned with Jarque and Villadelprat's open question but could not be dealt with in [16].

2. Preliminaries

System (1.1) with (1.2) can be expressed as

$$\dot{x} = -y - \sum_{\substack{i+j=m \\ i,j\ge 0}}^{n} j a_{i,j} x^i y^{j-1}, \qquad \dot{y} = x + \sum_{\substack{i+j=m \\ i,j\ge 0}}^{n} i a_{i,j} x^{i-1} y^j, \tag{2.1}$$

which has a center at the origin. As given in [17], the Period Coefficient Lemma implies that the minimum period $P(r_0, \lambda)$ of the periodic orbit $C(r_0)$ around the origin through a nonzero point $(r_0, 0)$ can be presented as

$$P(r_0,\lambda) = 2\pi + \sum_{k=1}^{\infty} p_k(\lambda) r_0^k, \tag{2.2}$$

where $\lambda = (a_{m,0}, \ldots, a_{0,n})$ and the $p_k(\lambda)$'s are called *period coefficients*. As indicated in [17], the center is *isochronous* if $p_k(\lambda) = 0$ for all $k \geq 1$. Therefore, the main task of this paper is to find a non-vanished period coefficient.

In order to compute those coefficients, we rewrite system (2.1) in the polar coordinates $x = r\cos\theta$, $y = \sin\theta$ as

$$\dot{r} = \sum_{k=m-1}^{n-1} G_k(\theta) r^k, \qquad \dot{\theta} = 1 + \sum_{k=m-2}^{n-2} S_k(\theta) r^k, \tag{2.3}$$

where

$$\begin{aligned} G_k(\theta) &= \sum_{i=0}^{k+1} a_{k+1-i,i}\{(k+1-i)\cos^{k-i}\theta \sin^{i+1}\theta - \mathrm{i}\cos^{k+2-i}\theta\sin^{i-1}\theta\}, \\ S_k(\theta) &= (k+2)\sum_{i=0}^{k+2} a_{k+2-i,i}\cos^{k+2-i}\theta\sin^i\theta. \end{aligned} \tag{2.4}$$

Note that the $a_{i,j}$'s $(i,j \geq 0)$ and $S_k(\theta)$'s are defined only for $m \leq i+j \leq n$ and $m-2 \leq k \leq n-2$ respectively in (1.2) and (2.3). For convenience in the latter computation, define in a complementary way

$$a_{i,j} = 0 \quad \forall\, i+j \leq m-1 \text{ or } i+j \geq n+1$$

and

$$S_k(\theta) = 0 \quad \forall\, k \leq m-3 \text{ or } k \geq n-1.$$