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$L(\varphi, \mu)$-averaging domains and Poincaré inequalities with Orlicz norms

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ABSTRACT

We characterize $L(\varphi, \mu)$-averaging domains using the Whitney covers and the quasihyperbolic metric and study the invariance of $L(\varphi, \mu)$-averaging domains under some mappings. As applications of the $L(\varphi, \mu)$-averaging domains, we prove the Poincaré inequality with Orlicz norms for solutions of the non-homogeneous A-harmonic equation in $L(\varphi, \mu)$-averaging domains.

1. Introduction

This paper deals with a generalization of the averaging domains, where the Lebesgue measure is replaced by a measure given by a weight function and also the space of bounded mean oscillation is taken in the Orlicz space setting. We first characterize $L(\varphi, \mu)$-averaging domains defined by the Orlicz norm in terms of the Whitney covers and the quasihyperbolic metric. Then, we show that $L(\varphi, \mu)$-averaging domains are invariant under quasi-isometrics. Finally, as applications of the $L(\varphi, \mu)$-averaging domains, we prove the global Poincaré inequality with Orlicz norms $\|u - u_{B_0}\|_{L(\varphi,\Omega,\mu)} \le C\|du\|_{L(\varphi,\Omega,\mu)}$ for solutions of the non-homogeneous A-harmonic equation in any bounded $L(\varphi, \mu)$-averaging domain Ω, which is an extension of the existing versions of the Poincaré inequalities with L^p-norms. Domains and mappings are closely related and widely applied in different fields, such as ordinary and partial differential equations, quasiregular mappings, potential theory and nonlinear elasticity, see [1–5]. Many interesting results have been established in studying different domains, mappings and their applications in recent years; see [6–8], for example.

As usual, we assume that Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 2$. Balls are denoted by B, and σB is the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. Also, we do not distinguish balls from cubes throughout this paper. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ a.e.. For a function u, we denote the average of u over B by $u_B = \frac{1}{|B|}\int_B u\,dx$.

In 1989, Staples introduced the following L^s-averaging domains in [3]. A proper subdomain $\Omega \subset \mathbb{R}^n$ is called an L^s-averaging domain, $s \ge 1$, if there exists a constant C such that

$$\left(\frac{1}{|\Omega|}\int_\Omega |u - u_\Omega|^s dx\right)^{1/s} \le C \sup_{B\subset\Omega}\left(\frac{1}{|\Omega|}\int_B |u - u_B|^s dx\right)^{1/s} \tag{1.1}$$

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for all $u \in L^s_{loc}(\Omega)$. Here the supremum is over all balls $B \subset \Omega$. Recently, the L^s-averaging domains were extended into the $L^s(\mu)$-averaging domains in [4]. It is very interesting to know that these two kinds of domains can be generalized, and the generalization is not trivial. This observation may have many applications in the future. See [8] for more results in averaging domains. Throughout this paper μ is a measure defined by $d\mu = w(x)dx$ and $w(x)$ is a weight. Now we consider the following $L(\varphi, \mu)$-averaging domains.

Definition 1.1. Let φ be a continuously increasing convex function on $[0, \infty)$ with $\varphi(0) = 0$ and let Ω be a domain with $\mu(\Omega) < \infty$. If u is a measurable function in Ω, then we define the Orlicz norm of u by

$$\|u\|_{L(\varphi,\Omega,\mu)} = \inf\left\{k > 0 : \frac{1}{\mu(\Omega)}\int_\Omega \varphi\left(\frac{|u(x)|}{k}\right) d\mu \le 1\right\}.$$

A continuously increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, is called an Orlicz function. A convex Orlicz function φ is often called a Young function. From Definition 1.1, it is easy to see that for any domain $\Omega \subset \mathbb{R}^n$

$$\frac{1}{\mu(\Omega)}\int_\Omega \varphi\left(\frac{|u(x)|}{\|u\|_{L(\varphi,\Omega,\mu)}}\right) d\mu \le 1 \tag{1.2}$$

if $\|u\|_{L(\varphi,\Omega,\mu)}$ is finite.

Definition 1.2. Let φ be a increasing convex function on $[0, \infty)$ with $\varphi(0) = 0$. We call a proper subdomain $\Omega \subset \mathbb{R}^n$ an $L(\varphi, \mu)$-averaging domain, if $\mu(\Omega) < \infty$ and there exists a constant C such that

$$\|u - u_{B_0,\mu}\|_{L(\varphi,\Omega,\mu)} \le C \sup_{B\subset\Omega} \|u - u_{B,\mu}\|_{L(\varphi,B,\mu)} \tag{1.3}$$

for some ball $B_0 \subset \Omega$ and all integrable functions u in Ω, where the measure μ is defined by $d\mu = w(x)dx$, $w(x)$ is a weight and the supremum is over all balls B with $B \subset \Omega$.

Definition 1.3. Let $\sigma > 1$. We say that w satisfies a weak reverse Hölder's inequality and write $w \in WRH(\Omega)$ when there exist constants $\beta > 1$ and $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w^\beta dx\right)^{1/\beta} \le C\frac{1}{|B|}\int_{\sigma B} w dx \tag{1.4}$$

for all balls B with $\sigma B \subset \Omega$.

In fact the space $WRH(\Omega)$ is independent of $\sigma > 1$, see [4].

Definition 1.4. The quasihyperbolic distance between x and y in Ω is given by

$$k(x, y) = k(x, y; \Omega) = \inf_\gamma \int_\gamma \frac{1}{d(z, \partial\Omega)} ds, \tag{1.5}$$

where γ is any rectifiable curve in Ω joining x to y, $d(z, \partial\Omega)$ is the Euclidean distance between z and the boundary of Ω.

Gehring and Osgood prove that for any two points x and y in Ω there is a quasihyperbolic geodesic arc joining them, see [1]. The quasihyperbolic metric provides a useful substitute for the hyperbolic metric. Applications can be found, for example, in [1,3,4]. In this paper we see that it also plays an important role in describing the $L(\varphi, \mu)$-averaging domains. The following theorem appears in [5].

Theorem 1.5. *Assume that $w \in WRH(\Omega)$. Let φ be a increasing convex function on $[0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) \le e^{bt}$ for some $0 \le b < \infty$ and $t \ge 1$. Then Ω is an $L(\varphi, \mu)$-averaging domain if and only if*

$$\int_\Omega \varphi(\alpha k(x, x_0)) d\mu < \infty$$

for each x_0 in Ω and some $\alpha > 0$.

Definition 1.6. We say that a weight w satisfies the A_r-condition, where $r > 1$, and write $w \in A_r(\Omega)$ when

$$\sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{1/(1-r)} dx\right)^{r-1} < \infty,$$

where the supremum is over all balls $B \subset \Omega$.

It is well known that $w \in A_r(\Omega)$ implies $w \in WRH(\Omega)$.

Definition 1.7. We call w a doubling weight and write $w \in D(\Omega)$ if there exists a constant C such that $\mu(2B) \le C\mu(B)$ for all balls B with $2B \subset \Omega$. If this condition holds only for all balls B with $4B \subset \Omega$, then w is weak doubling and we write $w \in WD(\Omega)$.