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Integrability of the coupled KdV equations derived from two-layer fluids: Prolongation structures and Miura transformations

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ABSTRACT

The Lax integrability of the coupled KdV equations derived from two-layer fluids [S.Y. Lou, B. Tong, H.C. Hu, X.Y. Tang, Coupled KdV equations derived from two-layer fluids, J. Phys. A: Math. Gen. 39 (2006) 513–527] is investigated by means of prolongation technique. As a result, the Lax pairs of some Painlevé integrable coupled KdV equations and several new coupled KdV equations are obtained. Finally, the Miura transformations and some coupled modified KdV equations associated with the Lax integrable coupled KdV equations are derived by an easy way.

1. Introduction

Nonlinear wave equations are widely used to describe complex phenomena in various sciences such as fundamental particle physics, plasma and fluid dynamics, statistical mechanics, protein dynamics, condensed matter, biophysics, nonlinear optics, quantum field theory, etc. [1,2]. One of the important tasks in mathematical physics is to test the integrability (or solvability) of nonlinear wave equations. During the past four decades the developments in nonlinear wave equations show that the completely integrable systems have very rich mathematical structures such as the existence of Lax pairs, bi-Hamiltonian structures, recursion operators and infinitely many generalized symmetries. In this work we assume that a system of equation is said to be integrable if it admits Lax pair.

It is well known that to an arbitrary given nonlinear wave equation it is very difficult to determine whether it can be associated with a Lax pair. One of the effective methods to test integrability is the prolongation structure (PS) method of Wahlquist and Estabrook [3], which was firstly proposed in 1975. In the past three decades, many researchers have developed this approach. For examples, Hermann [4] interpreted PS method as a connection in 1976; Morris [5] applied PS method to nonlinear evolution equations in $(2+1)$-dimensions for the first time; Fordy and Dodd [6,7] made the method more algorithmic and put it in an algebraic, instead of differential geometric framework, especially, they dealt with some quasi-polynomial flows; Guo [7] et al. proposed a covariant geometry theory for PS method; Deconinck [8] applied PS method to semi-discrete systems for the first time; and so on.

Recently, the general coupled KdV equations

$$\begin{aligned} &u_t+\delta_1 v u_x+2\delta_2 v v_x+\delta_3 (uv)_x+\delta_4 u_{xxx}+2\delta_5 u u_x=0,\\ &v_t+\epsilon_1 v u_x+2\epsilon_2 u u_x+\epsilon_3 (uv)_x+\epsilon_4 v_{xxx}+2\epsilon_5 v v_x=0, \end{aligned} \tag{1.1}$$

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where δ_i and ϵ_i ($i = 1, 2, \ldots, 5$) are arbitrary constants obtained from a two-layer fluids of the atmospheric dynamical system by Lou et al. [9] and some types of Painlevé-integrable coupled KdV equations were found for different selections of parameters.

In this work, by means of Dodd–Fordy algorithm of the Wahlquist–Estabrook prolongation technique, the 4×4 matrix spectral problems (Lax pairs) of some Painlevé integrable coupled KdV equations and several new coupled KdV equations are obtained. Finally, the Miura transformations and some coupled modified KdV equations associated with the Lax integrable coupled KdV equations are derived by an easy way.

2. The prolongation structures of the general coupled KdV equations

In this section, we will explore the prolongation structures of coupled KdV equations to construct their matrix spectral problems. Before doing this, we review a basic theorem in Lie algebra.

Theorem. *Let X and Y are two elements of Lie algebra $g = \mathrm{sl}(n+1, C)$ such that $[X, Y] = aY$, $(a \neq 0)$ and $X \in$ range ad Y. Then we may identity Y with $e_\pm$ and X with $\pm\frac{1}{2}ah$, where $e_\pm$ are the nilpotent elements of g and h is the neutral element of g.*

Firstly, we define a new set of independent variables

$$p = u_x, \qquad q = u_{xx} = p_x, \qquad z = v_x, \qquad r = v_{xx} = z_x. \tag{2.1}$$

Then (1.1) can be represented by the set of two-forms as follows

$$\begin{aligned}
\alpha_1 &= \mathrm{d}u \wedge \mathrm{d}t - p\mathrm{d}x \wedge \mathrm{d}t,\\
\alpha_2 &= \mathrm{d}p \wedge \mathrm{d}t - q\mathrm{d}x \wedge \mathrm{d}t,\\
\alpha_3 &= \mathrm{d}v \wedge \mathrm{d}t - z\mathrm{d}x \wedge \mathrm{d}t,\\
\alpha_4 &= \mathrm{d}z \wedge \mathrm{d}t - r\mathrm{d}x \wedge \mathrm{d}t,\\
\alpha_5 &= \mathrm{d}u \wedge \mathrm{d}x + (\delta_1 vp + 2\delta_2 vz + \delta_3 pv + \delta_3 uz + 2\delta_5 up)\mathrm{d}t \wedge \mathrm{d}x - \delta_4 \mathrm{d}q \wedge \mathrm{d}t,\\
\alpha_6 &= \mathrm{d}v \wedge \mathrm{d}x + (\epsilon_1 vp + 2\epsilon_2 up + \epsilon_3 pv + \epsilon_3 uz + 2\epsilon_5 vz)\mathrm{d}t \wedge \mathrm{d}x - \epsilon_4 \mathrm{d}r \wedge \mathrm{d}t,
\end{aligned}$$

which is closed under exterior differential, that is

$$\mathrm{d}\alpha_i = \sum_{j=1}^{6} g_j \wedge \alpha^j$$

where g_j ($j = 1, 2, \ldots, 6$) are differential 1-form.

The next step is to introduce the system of one-forms

$$\omega^i = \mathrm{d}y^i - F^i(u, v, p, q, z, r, y^i)\mathrm{d}x - G^i(u, v, p, q, z, r, y^i)\mathrm{d}t,$$

where y^i ($i = 1, 2, \ldots, n$) are called pseudopotentials and we assume F^i and G^i are of the form

$$F^i = F^i_j y^j, \qquad G^i = G^i_j y^j.$$

In what follows, we write F^i_j to be F and G^i_j to be G for simplicity.

We need $I \bigcup \{\omega^i\}$ to be a closed ideal, namely

$$\mathrm{d}\omega^i = \sum_{j=1}^{6} f^i_j \alpha^j + \eta^i \wedge \omega^i$$

which leads to a set of nonlinear partial differential equations as follows

$$\begin{aligned}
&F_p = F_q = F_z = F_r = 0, \qquad \delta_4 F_u + G_q = 0, \qquad \epsilon_4 F_v + G_r = 0,\\
&G_u p + G_p q + G_v z + G_z r + F_u[(\delta_1 + \delta_3)pv + 2\delta_2 vz + \delta_3 uz + 2\delta_5 up]\\
&\quad + F_v[(\epsilon_1 + \epsilon_3)pv + 2\epsilon_2 up + \epsilon_3 uz + 2\epsilon_5 vz] - [F, G] = 0,
\end{aligned} \tag{2.2}$$

where $[F, G] = FG - GF$.

One solution of this set of equations is

$$\begin{aligned}
F &= uX_1 + vX_2 + X_3, \qquad (2.3)\\
G &= -\delta_4 qX_1 - \epsilon_4 rX_2 + \delta_4 vpX_4 - \delta_4 pX_5 - \epsilon_4 uzX_4 - \epsilon_4 zX_6 + X_7 + u^2 X_8 + v^2 X_9 + uvX_{10} + uX_{11} + vX_{12},
\end{aligned}$$

with the integrability conditions

$$\begin{aligned}
&X_4 = [X_1, X_2], \qquad X_5 = [X_3, X_1], \qquad X_6 = [X_3, X_2], \qquad [X_3, X_7] = 0,\\
&[X_1, X_8] = [X_2, X_9] = [X_1, X_4] = [X_2, X_4] = 0, \qquad [X_1, X_7] + [X_3, X_{11}] = 0, \qquad (2.4)\\
&[X_2, X_7] + [X_3, X_{12}] = 0, \qquad [X_3, X_8] + [X_1, X_{11}] = 0, \qquad [X_1, X_9] + [X_2, X_{10}] = 0,\\
&[X_3, X_9] + [X_2, X_{12}] = 0, \qquad [X_1, X_{10}] + [X_2, X_8] = 0, \qquad [X_3, X_{10}] + [X_1, X_{12}] + [X_2, X_{11}] = 0,
\end{aligned}$$