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Computation of topological degree of unilaterally asymptotically linear operators and its applications☆

Xiaoying Liu *, Jingxian Sun

Department of Mathematics, Xuzhou Normal University, Xuzhou, 221116, Jiangsu, People's Republic of China

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ABSTRACT

Using the cone theory and the lattice structure, we establish some methods of computation of the fixed point index and the topological degree for the unilaterally asymptotically linear operators which are not assumed to be cone mappings. As applications, the existence results of boundary value problems for nonlinear elliptic partial differential equations are given.

1. Introduction

Let E be a Banach space with a cone P. Then E becomes an ordered Banach space under the partial ordering $\leq$ which is induced by P. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. P is called solid if it contains interior points, i.e. $\operatorname{int} P \neq \emptyset$. For the concepts and the properties about the cone we refer to [5–7]. For $w \in E$, Let $P_w = P + w = \{x \in E \mid x \geq w\}$ and $P^w = w - P = \{x \in E \mid x \leq w\}$.

Definition 1.1. Let E be a Banach space with a cone P, $A : E \to E$ a nonlinear operator. We call that A is a unilaterally asymptotically linear operator along P_w, if there exists a bounded linear operator L such that

$$\lim_{x \in P_w, \|x\| \to \infty} \frac{\|Ax - Lx\|}{\|x\|} = 0. \tag{1.1}$$

L is said to be the derived operator of A along P_w and will be denoted by A'_{P_w}.

Similarly, we can also define a unilaterally asymptotically linear operator along P^w.

Remark 1.1. In [5], the concept of the asymptotically linear operator A along P was defined (A is also said to be Frechét differentiable at ∞ along P), and A is needed to be a cone mapping, i.e. $A(P) \subset P$. If $w = 0$ in Definition 1.1, A is a unilaterally asymptotically linear operator along P and it is not assumed to be a cone mapping.

In the paper [10], using the partial ordering relation and the lattice structure, we established some theorems about computation of the topological degree for nonlinear operators which are not cone mappings. In this paper, we shall give some methods of computation of the topological degree for the unilaterally asymptotically linear operators which are not cone

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* Corresponding author.
E-mail addresses: liuxy@xznu.edu.cn, xzliuxy2007@yahoo.com.cn (X. Liu), jxsun7083@sohu.com (J. Sun).

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mappings. Using those new results, we investigate the boundary value problems for nonlinear elliptic partial differential equations.

We need the following lemmas.

Lemma 1.1 ([1]). *Let Ω be a bounded open set which contains θ and $A : \overline{\Omega} \to E$ a completely continuous operator. If*

$$Ax \neq \mu x, \quad \forall x \in \partial\Omega,\ \mu \geq 1,$$

then the topological degree $\deg(I - A, \Omega, \theta) = 1$.

One can prove the following conclusion using a similar method as in [1].

Lemma 1.2. *Let Ω be a bounded open set and $A : \overline{\Omega} \cap P_w \to P_w$ (or $A : \overline{\Omega} \cap P^w \to P^w$) a completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$, such that*

$$x - Ax \neq \mu u_0, \quad \forall\, x \in \partial\Omega \cap P_w,\ \mu \geq 0,$$
$$(\text{correspondingly } x - Ax \neq -\mu u_0, \quad \forall\, x \in \partial\Omega \cap P^w,\ \mu \geq 0)$$

then the fixed index $i(A,\ \Omega \cap P_w,\ P_w) = 0$ (correspondingly, $i(A,\ \Omega \cap P^w,\ P^w) = 0$).

Lemma 1.3 ([5]). *Let Ω be a bounded open set which contains θ and $A : \overline{\Omega} \to E$ a completely continuous operator with $A\theta = \theta$. Suppose that the Fréchet derivative A'_θ of A at θ exists, and 1 is not an eigenvalue of A'_θ, then there exists $r_0 > 0$, such that for $0 < r < r_0$,*

$$\deg(I - A,\ B_r,\ \theta) = (-1)^\beta,$$

where β is the sum of the algebraic multiplicities for all eigenvalues of A'_θ, lying in the interval $(1, \infty)$, and $B_r = \{x \in E \mid \|x\| < r\}$.

Lemma 1.4 ([5]). *Suppose that $A : E \to E$ is a completely continuous operator. If the Fréchet derivative A'_∞ of A at ∞ exists, and 1 is not an eigenvalue of A'_∞, then there exists $R_0 > 0$, such that for $R > R_0$,*

$$\deg(I - A,\ B_R,\ \theta) = (-1)^\beta,$$

where β is the sum of the algebraic multiplicities for all eigenvalues of A'_∞, lying in the interval $(1, \infty)$.

Let $L : E \to E$ be a bounded linear operator. L is said to be positive if $L(P) \subset P$. In this case, L is an increase operator, namely for $x,\ y \in E$, $x \leq y$ implies $Lx \leq Ly$. We have the following conclusion.

Lemma 1.5 ([10]). *Suppose that $L : E \to E$ is a positive bounded linear operator. If the spectral radius $r(L) < 1$, then $(I - L)^{-1}$ exists and is a bounded linear operator.*

2. Computation of the fixed point index for unilaterally asymptotically linear operators

Lemma 2.1. *Let $A : P_w \to E$ be a completely continuous operator. Suppose that A is a unilaterally asymptotically linear operator along P_w, and A'_{P_w} is the derived operator of A along P_w. Then*

(i) *$A'_{P_w} : P_w \to E$ is completely continuous, and therefore $A'_{P_w} : P_v \to E$ is completely continuous for any $v \in E$;*
(ii) *if there exists $u \in E$, such that $A(P_w) \subset P_u$, then $A'_{P_w}(P) \subset P$;*
(iii) *if 1 is not an eigenvalue of A'_{P_w} corresponding a positive eigenvector, then*

$$\inf_{x \in P, \|x\|=1} \{\|x - A'_{P_w} x\|\} = \sigma > 0. \tag{2.1}$$

Proof. Put $A'_{P_w} = L$.

(i) Let D be a bounded set in P_w. Then there exists $r > 0$ such that $\|x\| \leq r$ for all $x \in D$. If $L(D)$ is not relative compact, then there exist $\varepsilon_0 > 0$ and $\{x_n\} \subset D$, such that

$$\|Lx_n - Lx_m\| \geq \varepsilon_0, \quad n \neq m.$$

Let $y_n = x_n - w$, then $y_n \in P$ and

$$\|Ly_n - Ly_m\| = \|Lx_n - Lx_m\| \geq \varepsilon_0, \quad n \neq m. \tag{2.2}$$

Since L is bounded, it follows from (2.2) that $\inf\{\|y_n\|,\ n = 1, 2, \ldots\} = \alpha > 0$. By (1.1) in Definition 1.1, there exists $R_0 > 0$, such that

$$\|Ax - Lx\| < \frac{\varepsilon_0}{4(r + \|w\|)} \|x\|, \quad x \in P_w,\ \|x\| \geq R_0. \tag{2.3}$$