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Some notes on a nonlinear degenerate parabolic equation☆

Wen-Shu Zhou

Department of Mathematics, Dalian Nationalities University, 116600, China

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ABSTRACT

In this note we study nonexistence and long time behavior of solutions for a nonlinear degenerate parabolic equation of non-divergence type. In addition, we also construct some special explicit solutions which may blow up in finite time.

1. Introduction

In this note we study nonexistence and long time behavior of nonnegative solutions for the degenerate parabolic equation

$$\frac{\partial u}{\partial t} = u\,\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \gamma|\nabla u|^p \quad \text{in } \Omega_\infty = \Omega \times (0,+\infty), \tag{1}$$

with the Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ is a bounded domain with appropriately smooth boundary $\partial\Omega$, $p > 1$, $\gamma > 0$ and $u_0 \geqslant 0$.

We refer the reader to [1] and [2] for the motivation and references concerning the study of (1).

Since (1) may be degenerate at points where $u = 0$ or $|\nabla u| = 0$, we consider its weak solutions.

Definition 1.1. A nonnegative function u is called a weak solution of problem (1)–(3) if, for any $T > 0$, the following conditions are satisfied:

(a) $u \in L^\infty(\Omega_T) \cap L^p(0,T;W_0^{1,p}(\Omega))$ with $(u^\mu)_t \in L^2(\Omega_T)$;

(b) for any $\varphi \in C_0^\infty(\Omega_T)$, there holds

$$\iint_{\Omega_T}\left(-u\frac{\partial\varphi}{\partial t} + u|\nabla u|^{p-2}\nabla u\nabla\varphi + (1-\gamma)|\nabla u|^p\varphi\right)\mathrm{d}x\mathrm{d}t = 0;$$

(c) $\lim_{t\to 0^+}\int_\Omega |u^\mu(t) - u_0^\mu|\mathrm{d}x = 0$.

Here $\Omega_T = \Omega \times (0,T)$, $\mu = p\gamma/[2(p-1)] + 1/2$.

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E-mail address: wolfzws@163.com.

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The following existence theorem can be found in [1] for $p \geqslant 2$ and [2] for $1 < p < 2$.

Theorem 1.1. *Let $p > 1$, $0 < \gamma < 1$, and let $0 \leqslant u_0 \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$. Then problem (1)–(3) admits a weak solution.*

In the present paper, by a standard energy method, we first obtain the following results on nonexistence and long time behavior of solutions.

Theorem 1.2. *Let $p > 2$, and let $0 \leqslant u_0$, $0 < A \equiv \|u_0\|_{L^{1+\mu}(\Omega)} < \infty$. Then:*

(a) *If $\gamma > \frac{3(p-1)}{p-2}$, then problem (1)–(3) has no weak solutions.*
(b) *If $0 < \gamma \leqslant \frac{3(p-1)}{p-2}$, and if u is a weak solution of problem (1)–(3), then we have*

$$\|u(t)\|_{L^{1+\mu}(\Omega)} \leqslant (Ct + A^{1-p})^{-1/(p-1)}, \quad 0 < \gamma < \frac{3(p-1)}{p-2}, \tag{4}$$

where C is a positive constant independent of t, and

$$\|u(t)\|_{L^{1+\mu}(\Omega)} = \|u_0\|_{L^{1+\mu}(\Omega)}, \quad t > 0, \gamma = \frac{3(p-1)}{p-2}. \tag{5}$$

Remark 1.1. From Theorem 1.1 and (a) of Theorem 1.2, we see that if $0 < \gamma < 1$, then problem (1)–(3) admits a weak solution, and that if $\gamma > \frac{3(p-1)}{p-2}$ with $p > 2$, then problem (1)–(3) has no weak solutions. In the case where $1 \leqslant \gamma \leqslant \frac{3(p-1)}{p-2}$ with $p > 2$, we think that there should be global existence. Unfortunately, restricted by mathematical techniques and methods, we cannot prove this.

The result (b) of Theorem 1.2 shows that if $0 < \gamma < \frac{3(p-1)}{p-2}$, $\|u(t)\|_{L^{1+\mu}(\Omega)}$ decays with the rate $(Ct + A^{1-p})^{-1/(p-1)}$, and that if $\gamma = \frac{3(p-1)}{p-2}$, $\|u(t)\|_{L^{1+\mu}(\Omega)}$ does not decay as $t \to +\infty$.

If $0 < \gamma \leqslant (p-1)/p$, then $\mu \leqslant 1$. Clearly, any weak solution u of problem (1)–(3) satisfies $\frac{\partial u}{\partial t} \in L^2(\Omega_T)$ for any $T > 0$ if $0 < \gamma \leqslant (p-1)/p$. For $\gamma \geqslant 1$, however, this is not valid in general, which can be shown by the following two theorems.

Theorem 1.3. *Let $p > 1$, and let $0 \leqslant u_0$, $0 < A \equiv \|u_0\|_{L^1} < \infty$. Then:*

(a) *If $\gamma > 1$, then any weak solution u of problem (1)–(3) does not satisfy $\frac{\partial u}{\partial t} \in L^2(\Omega_T)$ for any $T \geqslant 1$ if A is sufficiently large.*
(b) *If $0 < \gamma \leqslant 1$, and if u with $\frac{\partial u}{\partial t} \in L^2(\Omega_T)$ for some $T > 0$ is a weak solution of problem (1)–(3), then*

$$\|u(t)\|_{L^1(\Omega)} \leqslant (Ct + A^{1-p})^{-1/(p-1)}, \quad 0 < t < T, 0 < \gamma < 1, \tag{6}$$

where C is a positive constant independent of T, and

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}, \quad 0 < t < T, \gamma = 1. \tag{7}$$

Theorem 1.4. *Let $p > 2$, $\gamma = 1$, and let $0 \leqslant u_0$, $0 < A \equiv \|u_0\|_{L^{\frac{4p-3}{2(p-1)}}(\Omega)} \mathrm{d}x < \infty$. Then any weak solution u of problem (1)–(3) does not satisfy $\frac{\partial u}{\partial t} \in L^2(\Omega_T)$ for sufficiently large T.*

In addition, we also construct some special explicit solutions of (1).

Let $\alpha > 0$, $\beta \geqslant 0$, $x_0 \in \mathbb{R}^N$, and define

$$T_\alpha = \alpha^{-1}\left(N + \frac{p\gamma}{p-1}\right)^{-1},$$

$$W_\alpha(t) = 1 - \alpha\left(N + \frac{p\gamma}{p-1}\right)t, \quad t \geqslant 0,$$

$$\psi_0(x) = \frac{p-1}{p}|x - x_0|^{p/(p-1)}, \quad x \in \mathbb{R}^N,$$

and

$$U_{\alpha\beta}(x, t; x_0) = \left[\frac{\alpha}{(p-1)W_\alpha}\right]^{1/(p-1)} \psi_0 + \beta W_\alpha^{-N/(p\gamma+(p-1)N]} \quad \text{in } \Omega_{T_\alpha}.$$

Then we have:

Theorem 1.5. *For any $\alpha > 0$, $\beta \geqslant 0$ and $x_0 \in \mathbb{R}^N$, $U_{\alpha\beta}(x, t; x_0)$ satisfies (1) in Ω_{T_α} and blows up at $t = T_\alpha$.*

Remark 1.2. By Theorem 1.5, for any $T > 0$, one can choose $\alpha = T^{-1}\left(N + \frac{p\gamma}{p-1}\right)^{-1}$ such that, for any $\beta \geqslant 0$ and $x_0 \in \mathbb{R}^N$, $U_{\alpha\beta}(x, t; x_0)$ blows up at $t = T$.

The proofs of Theorems 1.2–1.5 will be given in Section 2.