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A theorem on reversed S-shaped bifurcation curves for a class of boundary value problems and its application☆

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ABSTRACT

We study the bifurcation curves of positive solutions of the boundary value problem

$$\begin{cases} u''(x) + f_\varepsilon(u(x)) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $f_\varepsilon(u) = g(u) - \varepsilon h(u)$, $\varepsilon \in \mathbb{R}$ is a bifurcation parameter, and functions $g, h \in C[0, \infty) \cap C^2(0, \infty)$ satisfy five hypotheses presented herein. Assuming these hypotheses on fixed g and h, we prove that the bifurcation curve is reverse S-shaped on the $(\varepsilon, \|u\|_\infty)$-plane; that is, the bifurcation curve has *exactly two* turning points at some points $(\tilde{\varepsilon}, \|u_{\tilde{\varepsilon}}\|_\infty)$ and $(\varepsilon^*, \|u_{\varepsilon^*}\|_\infty)$ such that $\tilde{\varepsilon} < \varepsilon^*$ and $\|u_{\tilde{\varepsilon}}\|_\infty < \|u_{\varepsilon^*}\|_\infty$. In addition, we prove that $\varepsilon^* > 0$. Thus the exact number of positive solutions can be precisely determined by the values of $\tilde{\varepsilon}$ and ε^*. We give an application to the two-parameter bifurcation problem

$$\begin{cases} u''(x) + \lambda(1 + u^2 - \varepsilon u^3) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where λ, ε are two *positive* bifurcation parameters. Some new results are obtained.

1. Introduction

We study the bifurcation curves of positive solutions of the boundary value problem

$$\begin{cases} u''(x) + f_\varepsilon(u(x)) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

where

$$f_\varepsilon(u) = g(u) - \varepsilon h(u)$$

and $\varepsilon \in \mathbb{R}$ is a *real* bifurcation parameter. We first assume that functions $g, h \in C[0, \infty) \cap C^2(0, \infty)$ satisfy hypotheses (H1)–(H3):

(H1) $g(0) > 0$, $h(0) = 0$, $g(u), h(u) > 0$ for $u > 0$.

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(H2) $\lim_{u\to\infty} g(u)/u = \infty$. The positive function $g(u)/h(u)$ is strictly decreasing on $(0, \infty)$,

$$\lim_{u\to 0^+} \frac{g(u)}{h(u)} = \infty \quad \text{and} \quad \lim_{u\to\infty} \frac{g(u)}{h(u)} = 0.$$

(H3) $g''(u) > 0$ and $h''(u) > 0$ for $u > 0$. The positive function $g''(u)/h''(u)$ is strictly decreasing on $(0, \infty)$,

$$\lim_{u\to 0^+} \frac{g''(u)}{h''(u)} = \infty \quad \text{and} \quad \lim_{u\to\infty} \frac{g''(u)}{h''(u)} = 0.$$

Since $g, h \in C[0, \infty) \cap C^2(0, \infty)$ satisfy (H1) and (H2), for each $\varepsilon > 0$, there exists a unique positive number β_ε defined by

$$\frac{g(\beta_\varepsilon)}{h(\beta_\varepsilon)} = \varepsilon,$$

such that β_ε is a strictly decreasing continuous function of ε on $(0, \infty)$, $\lim_{\varepsilon\to 0^+} \beta_\varepsilon = \infty$ and $\lim_{\varepsilon\to\infty} \beta_\varepsilon = 0$. We have, for $\varepsilon > 0$,

$$f_\varepsilon(u) = g(u) - \varepsilon h(u) \begin{cases} > 0 & \text{for } 0 < u < \beta_\varepsilon, \\ = 0 & \text{for } u = \beta_\varepsilon, \\ < 0 & \text{for } u > \beta_\varepsilon, \end{cases} \tag{1.2}$$

and at $u = \beta_\varepsilon$,

$$f'_\varepsilon(\beta_\varepsilon) = g'(\beta_\varepsilon) - \varepsilon h'(\beta_\varepsilon) = g'(\beta_\varepsilon) - \frac{g(\beta_\varepsilon)}{h(\beta_\varepsilon)} h'(\beta_\varepsilon) = h(\beta_\varepsilon) \left(\frac{g(\beta_\varepsilon)}{h(\beta_\varepsilon)}\right)' \in (-\infty, 0) \tag{1.3}$$

by (H1) and (H2), and

$$f''_\varepsilon(\beta_\varepsilon) = g''(\beta_\varepsilon) - \varepsilon h''(\beta_\varepsilon) \in (-\infty, \infty).$$

Similarly, since $g, h \in C[0, \infty) \cap C^2(0, \infty)$ satisfy (H1) and (H3), for each $\varepsilon > 0$, there exists a unique positive number $\gamma_\varepsilon < \beta_\varepsilon$ defined by

$$\frac{g''(\gamma_\varepsilon)}{h''(\gamma_\varepsilon)} = \varepsilon, \tag{1.4}$$

such that γ_ε is a strictly decreasing continuous function of ε on $(0, \infty)$, $\lim_{\varepsilon\to 0^+} \gamma_\varepsilon = \infty$ and $\lim_{\varepsilon\to\infty} \gamma_\varepsilon = 0$. We have, for $\varepsilon > 0$,

$$f''_\varepsilon(u) = g''(u) - \varepsilon h''(u) \begin{cases} > 0 & \text{for } 0 < u < \gamma_\varepsilon, \\ = 0 & \text{for } u = \gamma_\varepsilon, \\ < 0 & \text{for } \gamma_\varepsilon < u < \beta_\varepsilon. \end{cases} \tag{1.5}$$

So by the above analysis, for $\varepsilon > 0$, $f_\varepsilon(u) \in C[0, \beta_\varepsilon] \cap C^2(0, \beta_\varepsilon]$. Meanwhile for $\varepsilon \le 0$, it is easy to see that $f_\varepsilon(u) = g(u) - \varepsilon h(u) > 0$ on $(0, \infty)$ and $f_\varepsilon(u) \in C[0, \infty) \cap C^2(0, \infty)$, and we define $\beta_\varepsilon = \infty$.

In this paper, for $\varepsilon \in \mathbb{R}$, we define

$$F_\varepsilon(u) = \int_0^u f_\varepsilon(t)\mathrm{d}t, \qquad G(u) = \int_0^u g(t)\mathrm{d}t, \qquad H(u) = \int_0^u h(t)\mathrm{d}t, \tag{1.6}$$

$$T_\varepsilon(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha [F_\varepsilon(\alpha) - F_\varepsilon(u)]^{-1/2}\, \mathrm{d}u \quad \text{for } 0 < \alpha < \beta_\varepsilon \begin{cases} < \infty & \text{if } \varepsilon > 0, \\ = \infty & \text{if } \varepsilon \le 0, \end{cases} \tag{1.7}$$

$$\theta_{f_\varepsilon}(u) = 2F_\varepsilon(u) - uf_\varepsilon(u), \qquad \theta_g(u) = 2G(u) - ug(u), \qquad \theta_h(u) = 2H(u) - uh(u). \tag{1.8}$$

In addition to (H1)–(H3), we assume that functions g and h satisfy hypotheses (H4) and (H5):

(H4) There exists a positive number $\bar{\varepsilon}$ such that

$$\theta_{f_\varepsilon}(\gamma_\varepsilon) = \theta_g(\gamma_\varepsilon) - \varepsilon\theta_h(\gamma_\varepsilon) < 0 \quad \text{for } 0 < \varepsilon < \bar{\varepsilon}.$$

(H5) There exists a positive number $\eta_0 < \beta_{\bar{\varepsilon}}$ such that

$$T_{\bar{\varepsilon}}(\eta_0) = 1, \qquad T_{\bar{\varepsilon}}(\alpha) > 1 \quad \text{for } \eta_0 < \alpha < \beta_{\bar{\varepsilon}},$$

and

$$\theta'_{f_\varepsilon}(u) > 0 \quad \text{on } (0, \min(\beta_\varepsilon, \eta_0)) \text{ for } \varepsilon \ge \bar{\varepsilon}.$$