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Strong convergence theorems for a common zero of a countably infinite family of α-inverse strongly accretive mappings

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ABSTRACT

Let E be a real reflexive Banach space which has a uniformly Gâteaux differentiable norm. Assume that every nonempty closed convex and bounded subset of E has the fixed point property for nonexpansive mappings. Strong convergence theorems for approximation of a common zero of a countably infinite family of α-inverse strongly accretive mappings are proved. Related results deal with strong convergence of theorems to a common fixed point of a countably infinite family of strictly pseudocontractive mappings.

1. Introduction

Let E be a real normed linear space with dual E^*. We denote by J, the *normalized duality mapping* from E to 2^{E^*} defined by:

$$J(x) := \{f \in E^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\},$$

where $\langle ., .\rangle$ denotes the duality pairing. It is well known that if E is *smooth* then J is single-valued (see, eg., [18]). In the sequel, we shall denote the single-valued normalized duality mapping by j.

A mapping A with domain $D(A)$ and range $R(A)$ in E is called *Lipschitz* if there exists $L \geq 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in D(A)$ in particular; if $L = 1$ then A is called *nonexpansive*; if $0 < L < 1$, then A is called *a contraction*. A mapping A is called *accretive* if, for all $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0. \tag{1.1}$$

If E is a Hilbert space, accretive operators are also called *monotone*. An operator A is called *m-accretive* if it is accretive and $\mathcal{R}(I + rA)$, range of $(I + rA)$, is E for all $r > 0$; and A is said to satisfy the range condition if $cl(D(A)) \subseteq \mathcal{R}(I + rA)$, $\forall r > 0$. A is called *α-inverse strongly accretive* if, for all $x, y \in D(A)$ there exist $\alpha > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha\|Ax - Ay\|^2. \tag{1.2}$$

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Equivalently, (1.2) can be written in the form

$$\langle x - y - (Ax - Ay), j(x - y)\rangle \leq \|x - y\|^2 - \alpha \|Ax - Ay\|^2 . \tag{1.3}$$

Without loss of generality, we may assume that $\alpha \in (0, 1)$. Observe that from (1.2), we have that α-inverse strongly accretive mapping A is Lipschitzian with Lipschitz constant $L = \frac{1}{\alpha}$.

It is now well known (see e.g. [23]) that if A is accretive then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems.

Consequently, considerable research works have been devoted to the approximation of zeros of accretive and α-inverse strongly accretive mappings (see, e.g., [2,3,6,7,9,12,13,15,16,19] and the references contained therein).

Interest in α-inverse strongly accretive mappings stems mainly from their firm connection with the important class of nonlinear *strictly pseudocontractive mappings* where a mapping T with domain $D(T)$ and range in E is called *strictly pseudocontractive* in the sense of Browder and Petryshyn [2] with constant α if $A := (I - T)$ is α-inverse strongly accretive, i.e., if for all $x, y \in D(T)$ there exist $\alpha > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle (x - Tx) - (y - Ty), j(x - y)\rangle \geq \alpha \|x - y - (Tx - Ty)\|^2 . \tag{1.4}$$

If $E = H$, a Hilbert space, (1.4) is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad k = (1 - \alpha) < 1, \tag{1.5}$$

and hence the class of strictly pseudocontractive mappings contains, as a subclass, the important class of nonexpansive mappings.

Let $N(A) := \{x \in D(A) : Ax = 0\}$ and $F(T) := \{x \in D(T) : Tx = x\}$ denote the null space of A and the fixed point set of T, respectively. Clearly, a zero of α-inverse strongly accretive mapping A is a fixed point of strictly pseudocontractive mapping $T := I - A$. For details on the subject, we refer the reader to [1,18].

Let K be a closed convex subset of a uniformly convex Banach space E and $T : K \to K$ be a *nonexpansive* mapping. In [8], Kirk introduced an iterative process given by

$$x_{n+1} := a_0 x_n + a_1 T x_n + a_2 T^2 x_n + \cdots + a_r T^r x_n, \tag{1.6}$$

where $a_i \geq 0, a_0 > 0$ and $\sum_{i=0}^{r} a_i = 1$, for approximating fixed points of nonexpansive mappings. Maiti and Saha [11] extended the results of Kirk [8].

Let K be a nonempty closed convex and *bounded* subset of a real Banach space E. Let $T_i : K \to K (i = 1, 2, \ldots, r)$ be nonexpansive mappings and let

$$S := a_0 I + a_1 T_1 + a_2 T_2 + \cdots + a_r T_r, \tag{1.7}$$

where $a_i \geq 0, a_0 > 0$ and $\sum_{i=0}^{r} a_i = 1$. Mappings T_i $(i = 1, 2, \ldots, r)$ with nonempty common fixed point set $D := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$, in K are said to satisfy *condition (A)* (see, e.g., [10,11,17]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$, such that $\|x - Sx\| \geq f(d(x, D))$ for all $x \in K$, where $d(x, D) := \inf\{\|x - z\| : z \in D\}$.

Recently, Liu et al. [10] introduced an iteration process

$$x_{n+1} := Sx_n, \tag{1.8}$$

where $x_0 \in K$, and showed that $\{x_n\}$ defined by (1.8) converges to a common fixed point of $\{T_i, i = 1, 2, \ldots, r\}$ in Banach spaces, *provided that* T_i $(i = 1, 2, \ldots, r)$ *satisfy condition (A)*. The result improves the corresponding results of Kirk [8], Maiti and Saha [11], Sentor and Dotson [17] and those of a whole host of other authors. But the assumption that T_i $(i = 1, 2, \ldots, r)$ satisfy condition (A) is strong. In 2004, Chidume et al. [5] proved convergence of $\{x_n\}$ without the requirement that $T_i (i = 1, 2, \ldots r)$ satisfy condition (A) in Banach spaces.

Recently, Yao et al. [20] proved strong convergence of the scheme $x_{n+1} := \lambda_{n+1} f(x_n) + \beta x_n + (1 - \beta - \lambda_{n+1}) W_n x_n, \forall n \geq 0$, where f is contractive self-mapping on K, $\{\lambda_n\}$ is a sequence in $(0, 1)$, β is a constant in $(0, 1)$ and W_n is the W mapping (see, e.g., [20]), to a countably infinite family of nonexpansive mappings in a Banach spaces.

In the case when the mappings are *α-inverse strongly accretive or accretive* mappings, finding a common zero point of a finite family of α-inverse strongly accretive or accretive mappings, the authors [21] proved the following.

Theorem ZS. *Let E be a strictly convex real reflexive Banach space which has uniformly Gáteaux differentiable norm and K be a nonempty closed convex subset of E. Assume that every nonempty closed convex and bounded subset of E has the fixed point property for nonexpansive mappings. Let $A_i : K \to E, i = 1, 2, \ldots, r$ be a family of m-accretive mappings with $\bigcap_{i=1}^{r} N(A_i) \neq \emptyset$. For given $u, x_1 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \theta_n u + (1 - \theta_n) S_r, \tag{1.9}$$

for all positive integers n, where $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$, with $J_{A_i} := (I + A_i)^{-1}$, called resolvent of A, for $0 < a_i < 1, i = 0, 2, \ldots, r$, $\sum_{i=0}^{r} a_i = 1$ and $\{\theta_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions: (i) $\lim_{n\to\infty} \theta_n = 0$; (ii) $\sum_{n=1}^{\infty} \theta_n = \infty$; (iii) $\sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty$; *or* (iii)* $\lim_{n\to\infty} \frac{|\theta_n - \theta_{n-1}|}{\theta_n} = 0$. *Then $\{x_n\}$ converges strongly to a common solution of the equation $A_i x = 0$ for $i = 1, 2, \ldots, r$.*