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# Approximation of the attainable sets of the nonlinear control systems with integral constraint on controls☆

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ABSTRACT

In this article the attainable sets of a control system are investigated. It is assumed that the behavior of the control system is described by a differential equation which is nonlinear with respect to the phase state vector and control vector. The admissible control functions are chosen from the closed ball centered at the origin with radius $\mu_0$ in $L_p([t_0, \theta]; \mathbb{R}^m)$ with $p \in (1, +\infty)$. An approximation method has been obtained for numerical construction of the attainable sets.



## 1. Introduction

One of the important constructions of the control systems theory is the attainable set notion. Attainable set is the set of all points to which the system can be steered at the instant of a given time and it is a very useful tool in the study of various problems of optimization, dynamical systems and differential game theory (see, e.g. [1,4,7,8,13,30,36]).

Numerous papers have been devoted to the study of various properties of the attainable sets of the control systems with geometric constraints on control (see, e.g. [1,4,10,14,20,30,37] and references therein). In papers [7,15,23] approximation and numerical construction methods of attainable sets of control systems with geometric constraints on control have been discussed.

Control problems with integral constraints on control arise in various problems of mathematical modeling. They naturally arise in the applications, such as control problems with bounded $L_p$ norms of the controls, control problems with prescribed bounded total energy of the trajectories and control systems with design uncertainties (see, e.g. [3,11,22,33,34,38] and references therein). For example, the motion of flying objects with variable mass, is described in the form of controllable system, where the control function has an integral constraint (see., e.g. [3,22,39]).

In [5,6,9,12,24,25,28,32] topological properties and numerical construction methods of the attainable sets of linear control systems with integral constraint on control functions are investigated. The attainable sets of affine control systems, i.e. the attainable sets of control systems which are nonlinear with respect to the phase state vector, but are linear with respect to the control vector have been considered in [16–18,29]. The properties of the attainable sets of the nonlinear control systems have been studied in [19,21,27,31,35].

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An approximation method for the construction of attainable sets of affine control systems with integral constraints on the control is given in [16,18]. In [29], using the topological properties of attainable sets of affine control systems, the continuity properties of minimum time and minimum energy functions are discussed.

The dependence of the attainable set on $p$ is studied in [17,19,25]. In [17], it is proved that attainable sets of affine control systems depend continuously on $p$. In [19], the same property is shown for nonlinear control systems. Closedness and various continuity properties of the attainable sets of the nonlinear control systems with integral constraint on controls are considered in [21].

In [31], it is proved that if the control resource is sufficiently small, then under some suitable assumptions on the right hand side of the system, the attainable set of the nonlinear control system with integral constraints on control is convex.

The value function of nonlinear optimal control problem with generalized integral constraints on control and phase state vectors is investigated in [27,35].

Consider the control system the behavior of which is described by the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) \in X_0 \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the phase state vector of the system, $u \in \mathbb{R}^m$ is the control vector, $t \in [t_0, \theta]$ is the time and $X_0 \subset \mathbb{R}^n$ is a compact set.

By $L_p([t_0, \theta], \mathbb{R}^m)$ $(p > 1)$ we denote the space of measurable functions $u(\cdot) : [t_0, \theta] \to \mathbb{R}^m$ with finite $\|u(\cdot)\|_p$ norm where $\|u(\cdot)\|_p = \left(\int_{t_0}^{\theta} \|u(t)\|^p \, dt\right)^{\frac{1}{p}}$ and $\|\cdot\|$ denotes the Euclidean norm.

For $p \in (1, \infty)$ and $\mu_0 > 0$ we set

$$U_p = \left\{u(\cdot) \in L_p\left([t_0, \theta], \mathbb{R}^m\right) : \|u(\cdot)\|_p \le \mu_0\right\} \tag{1.2}$$

where $\mu_0$ defines the resource of the control effort.

Every function $u(\cdot) \in U_p$ is said to be an admissible control function. It is obvious that the set of all admissible control functions $U_p$ is the closed ball centered at the origin with radius $\mu_0$ in $L_p([t_0, \theta]; \mathbb{R}^m)$.

It is assumed that the right hand side of the system (1.1) satisfies the following conditions:

1.A. The function $f(\cdot) : [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous;

1.B. For any bounded set $D \subset [t_0, \theta] \times \mathbb{R}^n$ there exist constants $L_1 = L_1(D) > 0$, $L_2 = L_2(D) > 0$ and $L_3 = L_3(D) > 0$ such that

$$\|f(t, x_1, u_1) - f(t, x_2, u_2)\| \le [L_1 + L_2(\|u_1\| + \|u_2\|)] \|x_1 - x_2\| + L_3 \|u_1 - u_2\|$$

for any $(t, x_1) \in D$, $(t, x_2) \in D$, $u_1 \in \mathbb{R}^m$ and $u_2 \in \mathbb{R}^m$;

1.C. There exists a constant $c > 0$ such that

$$\|f(t, x, u)\| \le c(1 + \|x\|)(1 + \|u\|)$$

for every $(t, x, u) \in [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m$.

Note that if the norm of the control effort is large then conditions 1.B and 1.C permit the system to assume high velocities.

Let the right hand side of the system (1.1) be affine, i.e. $f(t, x, u) = \varphi(t, x) + B(t, x)u$ and the functions $\varphi(\cdot) : [t_0, \theta] \times \mathbb{R}^n \to \mathbb{R}^n$, $B(\cdot) : [t_0, \theta] \times \mathbb{R}^n \to \mathbb{R}^m$ satisfy the assumptions given in [16–18,29]:

1.D. The functions $f(t, x)$ and $B(t, x)$ are continuous on $(t, x)$ and for any bounded set $D \subset [t_0, \theta] \times R^n$ there exist Lipschitz constants $\kappa_i = \kappa_i(D) \in (0, \infty)$ $(i = 1, 2)$ such that

$$\|\varphi(t, x_2) - \varphi(t, x_1)\| \le \kappa_1 \|x_2 - x_1\|, \qquad \|B(t, x_2) - B(t, x_1)\| \le \kappa_2 \|x_2 - x_1\|$$

for any $(t, x_1) \in D$, $(t, x_2) \in D$.

1.E. There exist constants $\nu_i \in (0, \infty)$ $(i = 1, 2)$ such that

$$\|\varphi(t, x)\| \le \nu_1(1 + \|x\|), \qquad \|B(t, x)\| \le \nu_2(1 + \|x\|)$$

for every $(t, x) \in [t_0, \theta] \times R^n$.

Then it is possible to verify that the function $f(t, x, u) = \varphi(t, x) + B(t, x)u$ satisfies conditions 1.A, 1.B and 1.C.

Let $u_*(\cdot) \in U_p$. The absolutely continuous function $x_*(\cdot) : [t_0, \theta] \to \mathbb{R}^n$ which satisfies the equation $\dot{x}_*(t) = f(t, x_*(t), u_*(t))$ a.e. in $[t_0, \theta]$ and the initial condition $x_*(t_0) = x_0 \in X_0$ is said to be a solution of the system (1.1) with initial condition $x_*(t_0) = x_0$, generated by the admissible control function $u_*(\cdot)$. By the symbol $x(\cdot; t_0, x_0, u(\cdot))$ we denote the solution of the system (1.1) with initial condition $x(t_0) = x_0$, which is generated by the admissible control function $u(\cdot)$. Note that conditions 1.A–1.C guarantee the existence, uniqueness and extendability of the solutions up to the instant of time $\theta$ for every given $u(\cdot) \in U_p$ and $x_0 \in X_0$.

Let us define the sets

$$Y_p(t_0, X_0) = \left\{x(\cdot; t_0, x_0, u(\cdot)) : x_0 \in X_0, \ u(\cdot) \in U_p\right\},$$

$$X_p(t; t_0, X_0) = \left\{x(t) \in \mathbb{R}^n : x(\cdot) \in Y_p(t_0, X_0)\right\}$$

where $t \in [t_0, \theta]$.