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The best uniform polynomial approximation to class of the form $\frac{1}{(a^2 \pm x^2)}$

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ABSTRACT

Using Chebyshev expansion properties we obtain explicitly the best uniform polynomial approximation out of P_{2n} to a class of rational functions of the form $1/(a^2 \pm x^2)$ on $[-c, c]$. In this way we give some new theorems about the best approximation of this class of rational functions. Furthermore we obtain the alternating set of this class of functions.

1. Introduction

One of the important and applicable subjects in approximation theory is the best L_m approximation problem. This problem is defined as in the following:

Definition 1 (*[1]*). Suppose P_n denotes the space of polynomials of degree at most n, then for given $f \in C[d, e]$, there exists a unique polynomial $p_n^* \in P_n$ such that

$$\left\| f - p_n^* \right\|_m \le \| f - p \|_m , \quad \forall\, p \in P_n, \tag{1}$$

we call p_n^* the best L_m polynomial approximation out of P_n to f on $[d, e]$. Also for the case L_∞ (uniform norm) we have:

$$E_n(f; [d, e]) = \max_{d \le x \le e} \left| f(x) - p_n^*(x) \right| < \max_{d \le x \le e} | f(x) - p(x) | , \quad \forall\, p \in P_n, \tag{2}$$

and in this case p_n^* is called the best uniform polynomial approximation to f on $[d, e]$.

A large number of papers and books have considered this problem in various viewpoints [1,2]. The main questions of this problem are existence, uniqueness and characterization of the solution. The existence and uniqueness of the solution of the best L_m approximation problem for $f \in C[d, e]$, are proved in [1,2]. In other way, there are theorems to characterize the solution of the best L_m approximation. These theorems characterize the solution for L_m $(1 \le m < \infty)$ norm explicitly in general case (for all smooth functions). But for L_∞ norm (uniform norm) the characterization theorem (Chebyshev's alternation theorem [3]) does not provide the solution of the best uniform polynomial approximation explicitly. Therefore the researchers in this field obtain the characterization of the best uniform polynomial approximation for special classes of functions. Furthermore several of these researches were focused on classes of functions possessing a certain expansion by

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Chebyshev polynomials. For example [2] Chebyshev characterized the best uniform polynomial approximation to a class of rational function $1/(x-a)$ where $a > 1$. In this manner Jokar and Mehri in [4] studied this class of rational function. Also Lubinsky in [5] showed that Lagrange interpolants at the Chebyshev zeros yield the best relevant polynomial approximation of $1/(1+(ax)^2)$ on $[-1, 1]$. Furthermore Newman and Rivlin [6] proved that if f is of the form:

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x), \tag{3}$$

where a_k are positive, increasing and satisfy $a_k \le a_{k-1}a_{k+1}$, $k \ge 1$, then we have:

$$E_n(f) \le \sum_{k=n+1}^{\infty} a_k \le 4eE_n(f). \tag{4}$$

Bernstein [7] showed that if $f(x)$ is a continuous function of period 2π with the Fourier series

$$f(x) = \sum_{k=0}^{\infty} a_k \cos n_k x, \tag{5}$$

subject to the conditions that $a_k > 0$ and $\frac{n_{k+1}}{n_k} = 2p_k + 1$, where p_k is a positive integer, then we have:

$$E_n(f) = \sum_{k=n+1}^{\infty} a_k, \tag{6}$$

where $(n_j \le n \le n_j + 1)$ (see also [2]).

In this paper using Chebyshev expansion we obtain the best uniform polynomial approximation out of P_{2n} to a class of rational functions of the form $1/(a^2 \pm x^2)$ on $[-c, c]$. In the following we give some definitions and theorems. First we state characterization of the best uniform polynomial approximation out of P_n via the following theorem:

Theorem 1 (*Chebyshev Alternation Theorem: [3]*). *Suppose $f \in C[d, e]$, and $\varepsilon(x) = f(x) - p_n(x)$. Then p_n is the best uniform approximation p_n^* to f on $[d, e]$ if and only if there exist at least $n+2$ points $x_1 < x_2 < \cdots < x_{n+2}$ in $[d, e]$, for which $|\varepsilon(x_i)| = \max_{d\le x\le e} |f(x) - p_n(x)|$, with $\varepsilon(x_{i+1}) = -\varepsilon(x_i)$.*

Definition 2. The Chebyshev polynomial in $[-1, 1]$ of degree n is denoted by T_n and is defined by $T_n(x) = \cos(n\theta)$, where $x = \cos(\theta)$.

Definition 3. The Chebyshev polynomial in $[d, e]$ of degree n is denoted by T_n^* and is defined by $T_n^*(x) = \cos(n\theta)$, where

$$\cos(\theta) = \frac{2x - (d+e)}{e-d}. \tag{7}$$

Note that $T_n(x)$ is [8] a polynomial of degree n with leading coefficient 2^{n-1}. Now before we start to determine the best uniform approximation polynomial to $1/(a^2 - x^2)$ we prove the following lemma as we need it in the next section:

Lemma 1. *For $x = \cos(\theta)$, $|t| < 1$ and p (natural number) we have:*

$$\text{(a)} \quad \sum_{j=0}^{\infty} t^{pj} T_{pj}(x) = \frac{1 - t^p\cos(p\theta)}{1 + t^{2p} - 2t^p\cos(p\theta)}, \tag{8}$$

$$\text{(b)} \quad \sum_{j=0}^{n-1} t^{pj} T_{pj}(x) = \frac{1 - t^p\cos(p\theta) - t^{pn}\cos(pn\theta) + t^{pn+p}\cos(pn\theta)\cos(p\theta) + t^{pn+p}\sin(pn\theta)\sin(p\theta)}{1 + t^{2p} - 2t^p\cos(p\theta)}. \tag{9}$$

Proof. The proof of this lemma can be easily obtained noting to $\mathrm{e}^{\mathrm{i}n\theta} = \cos(n\theta) + \mathrm{i}\sin(n\theta)$. □

2. Best approximation of $1/(a^2 - x^2)$

In this section we determine the best uniform polynomial approximation out of P_{2n} to $1/(a^2 - x^2)$ on $[-1, 1]$ using the results given in the previous section.

Theorem 2. *The best uniform polynomial approximation out of P_{2n} to $1/(a^2 - x^2)$ where $a^2 > 1$, on $[-1, 1]$, is*

$$p_{2n}^*(x) = \frac{4t^2}{t^4 - 1} - \frac{8t^2}{t^4 - 1}\sum_{k=0}^{n-1} t^{2k} T_{2k}(x) + \frac{8t^{2n+2}}{(t^4 - 1)^2} T_{2n}(x), \tag{10}$$