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Strong convergence theorems by the hybrid method for families of mappings in Banach spaces

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ABSTRACT

Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is Gâteaux differentiable and let $\{T_n\}$ be a family of mappings of C into itself such that the set of all common fixed points of $\{T_n\}$ is nonempty. We consider a sequence $\{x_n\}$ generated by the hybrid method in mathematical programming. And we give the conditions of $\{T_n\}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$ and generalize the results of [K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence theorems by the hybrid method for families of nonexpansive mappings in Hilbert spaces, Taiwanese J. Math. 10 (2006) 339–360; S. Ohsawa, W. Takahashi, Strong convergence theorems for resolvents of maximal monotone operators in Banach spaces, Arch. Math. 81 (2003) 439–445].

1. Introduction

Throughout this paper, let $\mathbf{N}$ and $\mathbf{R}$ be the set of all positive integers and the set of all real numbers, respectively. Haugazeau [6] introduced a sequence $\{x_n\}$ generated by the hybrid method, that is, let $\{T_n\}$ be a family of mappings of a real Hilbert space H into itself with $\cap_{n=0}^{\infty} F(T_n) \neq \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 = x \in H, \\ y_n = T_n x_n, \\ C_n = \{z \in H : (x_n - y_n, y_n - z) \geq 0\}, \\ Q_n = \{z \in H : (x_n - z, x - x_n) \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x) \end{cases} \tag{1}$$

for each $n \in \mathbf{N} \cup \{0\}$, where $(\cdot, \cdot)$ is an inner product and $P_{C_n \cap Q_n}$ is the metric projection onto $C_n \cap Q_n$. He proved a strong convergence theorem when $T_n = P_{n(\bmod m)+1}$ for every $n \in \mathbf{N} \cup \{0\}$, where P_i is the metric projection onto a nonempty closed convex subset C_i of H for each $i = 1, 2, \ldots, m$ and $\cap_{i=1}^{m} C_i \neq \emptyset$. Later, Solodov and Svaiter [17], Bauschke and Combettes [3] and Nakajo and Takahashi [10] studied the hybrid method in a Hilbert space (see also [1,7–9,11,12]) and authors [13] obtained the following unified result for the hybrid method: Let C be a nonempty closed convex subset of a real

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Hilbert space H and let $\{T_n\}$ be a family of mappings of C into itself with $F := \bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$ which satisfies the following condition: There exists $\{a_n\} \subset (-1, \infty)$ such that

$$\|T_n x - z\|^2 \leq \|x - z\|^2 - a_n \|(I - T_n)x\|^2$$

for every $n \in \mathbf{N} \cup \{0\}$, $x \in C$ and $z \in F(T_n)$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = T_n P_C(x_n + \varepsilon_n), \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n + \varepsilon_n - z\|^2 - a_n \|P_C(x_n + \varepsilon_n) - y_n\|^2\}, \\ Q_n = \{z \in C : (x_n - z, x - x_n) \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x) \end{cases} \tag{2}$$

for each $n \in \mathbf{N} \cup \{0\}$, where $\{\varepsilon_n\} \subset H$ and $\liminf_{n\to\infty} a_n > -1$. Then, the following hold:

(i) A sequence $\{x_n\}$ generated by (2) is well defined and $\{x_n\} \subset C$;
(ii) Assume that $\sum_{n=0}^{\infty} \|\varepsilon_n\|^2 < \infty$ and for every bounded sequence $\{z_n\}$ in C, $\sum_{n=0}^{\infty} \|z_{n+1} - z_n\|^2 < \infty$ and $\sum_{n=0}^{\infty} \|z_n - T_n z_n\|^2 < \infty$ imply $\omega_w(z_n) \subset F$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$. Then, $\{x_n\}$ converges strongly to $z_0 = P_F(x)$;
(iii) Assume that $\lim_{n\to\infty} \|\varepsilon_n\| = 0$ and for every bounded sequence $\{z_n\}$ in C, $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ implies $\omega_w(z_n) \subset F$. Then, $\{x_n\}$ converges strongly to $z_0 = P_F(x)$.

On the other hand, Ohsawa and Takahashi [14] first studied the hybrid method by the metric projection in a real Banach space and extended the result [17] for maximal monotone operators to a uniformly convex Banach space whose norm is Gâteaux differentiable.
Motivated by the result of [14], in this paper, we consider the generalization of [13] to a Banach space and prove strong convergence theorems. And using the results, we consider the convex feasibility problem.

2. Preliminaries and lemmas

Throughout this paper, let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. The value of $x^* \in E^*$ at a point $x \in E$ is denoted by $\langle x, x^* \rangle$. We write $x_n \rightharpoonup x$ to indicate that a sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize strong convergence. We define the modulus of convexity of E δ_E as follows: δ_E is a function of $[0, 2]$ into $[0, 1]$ such that $\delta_E(\varepsilon) = \inf\{1 - \|x + y\|/2 : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}$ for every $\varepsilon \in [0, 2]$. E is called uniformly convex if $\delta_E(\varepsilon) > 0$ for each $\varepsilon > 0$. It is known that if E is uniformly convex, then $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. Let $G = \{g : [0, \infty) \longrightarrow [0, \infty) : g(0) = 0, g : \text{continuous, strictly increasing, convex}\}$. We have the following theorem [20, Theorem 2] for a uniformly convex Banach space.

Lemma 2.1. *E is a uniformly convex Banach space if and only if for every bounded subset B of E, there exists $g_B \in G$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g_B(\|x - y\|)$$

for all $x, y \in B$ and $0 \leq \lambda \leq 1$

Remark 2.2. By Lemma 2.1, we know that E is uniformly convex if and only if there exists $G_0 \subset G$ such that for every bounded subset B of E, there exists $g_B \in G_0$ such that

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g_B(\|x - y\|)$$

for each $x, y \in B$ and $\lambda \in [0, 1]$. In a real Hilbert space H,

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$$

holds for each $x, y \in H$ and $\lambda \in \mathbf{R}$, so we can choose $G_0 = \{g(t) = t^2\}$

The norm of E is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for every $x, y \in S(E)$, where $S(E) = \{x \in E : \|x\| = 1\}$. Let J be the duality mapping from E into E^* defined by $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$ for each $x \in E$. It is known that the duality mapping J is single valued and $\|x\|^2 - \|y\|^2 \geq 2\langle x - y, J(y) \rangle$ holds for every $x, y \in E$ if E has a Gâteaux differentiable norm.

Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $x \in E$. Then, there exists a unique element $x_0 \in C$ such that $\|x_0 - x\| = \inf_{y\in C} \|y - x\|$. Putting $x_0 = P_C(x)$, we call P_C the metric projection onto C; see [5, p. 12]. And we have the following result [18, p. 196] for the metric projection.