Nonlinear Analysis 71 (2009) 860–875

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Minimization of a quasi-linear Ginzburg–Landau type energy

Rejeb Hadiji [a,*], Carmen Perugia [b]

[a] Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050, UFR des Sciences et Technologie, 61, Avenue du Général de Gaulle Bât. P3, 4e étage, 94010 Créteil Cedex, France

[b] Universitá del Sannio, Dipartimento di Studi Geologici ed Ambientali, Via Dei Mulini 59/A Palazzo Inarcassa, 82100, Benevento, Italy

ARTICLE INFO

Article history:
Received 13 September 2008
Accepted 3 November 2008

MSC:
35B25
35J55
35B40

Keywords:
Ginzburg–Landau equation
Quasi-linear problem
S^1 valued map

ABSTRACT

Let G be a smooth bounded domain in $\mathbb{R}^2$. Consider the functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G \left(p_0 + t\,|x|^k\,|u|^l\right)|\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G \left(1-|u|^2\right)^2$$

on the set $H^1_g(G,\mathbf{C}) = \left\{u \in H^1(G,\mathbf{C});\ u = g \text{ on } \partial G\right\}$ where g is a given boundary data with degree $d \geq 0$. In this paper we will study the behavior of minimizers u_ε of E_ε and we will estimate the energy $E_\varepsilon(u_\varepsilon)$.

1. Introduction

Let G be a bounded smooth domain of $\mathbf{R}^2$, $g : \partial G \to S^1$ a smooth boundary data of degree $d \geq 0$. For $\varepsilon > 0$, $p_0 > 0$, $t > 0$, $k \geq 2$ and $l \geq 2$ define the following functional of Ginzburg–Landau type

$$E_\varepsilon(u) = \frac{1}{2}\int_G \left(p_0 + t\,|x|^k\,|u|^l\right)|\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G \left(1-|u|^2\right)^2 \tag{1.1}$$

on the set

$$H^1_g(G,\mathbf{C}) = \left\{u \in H^1(G,\mathbf{C});\ u = g \text{ on } \partial G\right\}. \tag{1.2}$$

We shall understand that if $\int_G |u|^l|\nabla u|^2 = \infty$ then $E_\varepsilon(u) = \infty$. In this paper we are interested in the study of the asymptotic behavior of

$$\min_{u\in H^1_g(G,\mathbf{C})} E_\varepsilon(u). \tag{1.3}$$

When $t = 0$, $k = 0$ and $d = 0$, Bethuel et al. [1] showed that as ε tends to 0, u_ε tends to a harmonic u_0 which is equal to g on ∂G in $C^{1,\alpha}(\overline{G})$. It is easy to adapt the same method as in [1] to obtain the same result when $k \neq 0$.

The case when $t = 0$ and $d > 0$, corresponding to the Ginzburg–Landau energy, was studied by Bethuel et al. in [2] (see also Struwe [3]), where it was shown that:

* Corresponding author.
E-mail addresses: hadiji@univ-paris12.fr (R. Hadiji), perugia@unisannio.it (C. Perugia).

0362-546X/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2008.11.078

(i) for a subsequence $\varepsilon_n \to 0$ we have, $u_{\varepsilon_n} \to u_* = e^{i\phi} \prod_{j=1}^{d} \frac{z-a_j}{|z-a_j|}$ in $C^{1,\alpha}(\bar{G} \setminus \{a_1, \ldots, a_d\})$, where $a_1, \ldots, a_d$ are distinct points in G and ϕ is a smooth harmonic function determined by the requirement $u_* = g$ on ∂G.
(ii) $E_\varepsilon(u_\varepsilon) = 2\pi d|\log \varepsilon| + O(1)$ as $\varepsilon \to 0$.

In the case where $t > 0$ and $l = 0$, the problem for more general weight depending only on x is studied, see [6–8] and [4,5]. They showed that the presence of the weight forces the location of the vortices near the minima of the weight and when the degree is greater than the number of the minima of p the interaction between vortices, led to a term of order $\ln \ln \frac{1}{\varepsilon}$. It is also shown in the above references that the zeros of u_ε are located, for small ε, near the minima of the weight.

In this paper, we study the effect of the presence of $|u|$ in the weight $p_0 + t|x|^k s^l$. Our weight is a particular one and gives a significant situation. For instance, if we consider the case where $k = 0$, we show that we obtain a similar result of convergence as in [1] but the energy is greater than their energy. More precisely, in Theorem 1 we examine the case $\deg(g, \partial G) = 0$, $k \geq 0$ and $l \geq 0$. In Theorem 2 we examine the case $\deg(g, \partial G) > 0$, $k = 0$ and $l \geq 0$. In Theorem 3 we examine the most general case $\deg(g, \partial G) > 0$, $k \neq 0$ and $l \neq 0$. In both the last two cases we obtain a convergence result for a sequence of minimizers of our problem and we show that, under a small perturbation of the weight $p_0 + t|x|^k s^l$, the singularities of the limit problem are minima of $p_0 + t|x|^k$. As regards the energy, in Theorem 2, as $\varepsilon_n \to 0$ we get

$$E_{\varepsilon_n}(u_{\varepsilon_n}) = \pi d\,(p_0 + t) \ln \frac{1}{\varepsilon_n} + O\,(1) \tag{1.4}$$

while in Theorem 3 where, without loss of generality, we can suppose that $0 \in G$, as $\varepsilon_n \to 0$ we obtain

$$E_{\varepsilon_n}(u_{\varepsilon_n}) = \pi p_0 \left(d \ln \frac{1}{\varepsilon_n} + \frac{d^2 - d}{k} \ln \ln \frac{1}{\varepsilon_n} \right) + O\,(1)\,. \tag{1.5}$$

The motivation of our study for the functional (1.1) comes from type II superconductors in the presence of vortices, see [4–10].

The presence of the weight function is motivated by the problem of the pinning of vortices. It forces the location of the vortices to some favorite sites. In the case where $l = 0$ the regions where the weight is relatively small are called weak links, see [11]. So, we expect that the minima of the weight $p_0 + t|x|^k s^l$ will play an important role. As we shall show below, the zeros of a minimizer of our problem are located, for small ε, near the minima of $p_0 + t|x|^k$.

2. Setting of the problem and some preliminary results

At first, let us recall a definition and a lemma contained in [12].

Definition 1. Let Ω be an open set of $\mathbb{R}^p$, $1 \leq p \leq \infty$, $(\Omega, \Im, \mu)$ denote a measure space with μ non-negative and finite and $\Im$ is μ-complete. Set $\mathbf{B}_n$ the borel σ-field of $\mathbb{R}^n$. A function $f : \Omega \times \mathbf{R}^m \times \mathbf{R}^n \to]-\infty, +\infty]$ is said to be a normal-convex integrand if f is $\Im \otimes \mathbf{B}_m \otimes \mathbf{B}_n$-measurable function and there exists a μ-negligible set $N \subset \Omega$ such that

$$f\,(x, \cdot, \cdot) \text{ is l.s.c. on } \mathbf{R}^m \times \mathbf{R}^n \text{ for every } x \in \Omega - N$$
$$f\,(x, s, \cdot) \text{ is convex on } \mathbf{R}^n \text{ for every } x \in \Omega - N, \quad s \in \mathbf{R}^m.$$

Lemma 2.1. *Let Ω be a bounded open set of $\mathbf{R}^n$ with Lipschitz boundary and let $f : \Omega \times \mathbf{R}^m \times \mathbf{R}^{mn} \to [0, +\infty]$ be a normal-convex integrand in the sense of Definition 1. Then the functional*

$$F\,(u) = \int_\Omega f\,(x, u, \nabla u) \tag{2.1}$$

is sequentially weakly $W^{1,1}\left(\Omega, \mathbf{R}^m\right) - l.s.c.$

As a consequence of Lemma 2.1 we have

Lemma 2.2. *Let G be a bounded regular open set of $\mathbf{R}^2$. Then, the functionals*

$$F_2\,(u) = \int_G \left(p_0 + t\,|x|^k\,|u|^l\right) |\nabla u|^2 \tag{2.2}$$

and

$$F_1\,(u) = \int_G |x|^k\,|u|^l\,|\nabla u|^2 \tag{2.3}$$

are sequentially weakly $W^{1,1}\left(G, \mathbf{R}^2\right) - l.s.c.$