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The retractions onto the common fixed points of some families and semigroups of mappings

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ABSTRACT

Assume that C is a closed convex subset of a reflexive Banach space E and $\varphi = \{T_i\}_{i\in I}$ is a family of self-mappings on C of type (γ) such that $F(\varphi)$, the common fixed point set of φ, is nonempty. From our results in this paper, it can be derived that: (a) If $\cup_{i\in I} F(T_i)$ is contained in a 3-dimensional subspace of E then $F(\varphi)$ is a nonexpansive retract of C; (b) If φ is commutative, there exists a retraction R of type (γ) from C onto $F(\varphi)$, such that $RT_i = T_iR = R(\forall i)$, and every closed convex φ-invariant subset of C is R-invariant; the same result holds for a non-commutative right amenable semigroup φ, under some additional assumptions. Moreover, the existence of a (T_i)-ergodic retraction R of type (γ) from $\widetilde{C} = \{(x_i) \in l^\infty(E) : x_i \in C, \forall i \in I\}$ onto $F(\varphi)$ in $l^\infty(E)$ for the family φ is discussed. We also apply some of our results to find ergodic retractions for nonexpansive affine mappings.

1. Introduction

Let E be a Banach space and C a nonempty closed and convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$. A mapping T is said to be a retraction if $T^2 = T$. A subset F of C is called a nonexpansive retract of C if either $F = \varnothing$ or there exists a retraction of C onto F which is a nonexpansive mapping. In [9], Bruck initiated the study of the structure of the fixed point set $F(T) = \{x : Tx = x\}$ in a general Banach space E: If C is a locally weakly compact convex subset of E and $T : C \to C$ is nonexpansive and satisfies a conditional fixed point property, then $F(T)$ is a nonexpansive retract of C. The same author [4] used this fact to derive the existence of fixed points for a commuting family of nonexpansive mappings and obtained in [6] similar results for more general classes of mappings. Benavides and Ramirez [3] considered a weakly asymptotically nonexpansive mapping $T : C \to C$ and under some assumptions on C, showed that there is a nonexpansive retraction R from C onto $F(T)$ such that $RT = R$ and every closed convex T-invariant subset of C is also R-invariant. Following the terminology introduced by Bruck in [6], such a retraction R is called a T-ergodic retraction. The existence of nonexpansive retractions onto the common fixed points of a commuting family of nonexpansive mappings in a Banach space was investigated in [9,4]. The existence of retractions for non-commutative families of nonexpansive mappings for a strictly convex Banach space was studied in [7]; however, it is an open problem whether it can be extended to a general Banach space. For some related metric space results see, for example, [1,11,17,18]. On the other hand, the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space was established by Baillon [2]: Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesaro means $S_n(x) = \frac{1}{n}\sum_{k=1}^{n} T^k x$ converge weakly to some $y \in F(T)$. In Baillon's

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theorem, putting $y = Rx$ for each $x \in C$, R is a nonexpansive retraction of C onto $F(T)$ such that $RT^n = T^nR = R$ for all positive integers n and $Rx \in \overline{co}\{T^nx : n = 1, 2, \ldots\}$ for each $x \in C$. Takahashi [30] proved the existence of such retractions, "ergodic retractions", for non-commutative semigroups of nonexpansive mappings in a Hilbert space H: If S is an amenable semigroup, C is a closed, convex subset of H and $\varphi = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that the set $F(\varphi)$ of common fixed points of φ is nonempty, then there exists a nonexpansive retraction R from C onto $F(\varphi)$ such that $RT_t = T_tR = R$ for each $t \in S$ and $Rx \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$. These results were extended to a uniformly convex Banach space for commutative semigroups by Hirano, Kido and Takahashi [13], and after the works in [21,23], for non-commutative amenable semigroups by Lau, Shioji and Takahashi [22]. Furthermore, the existence of ergodic retractions for semigroups of nonexpansive mappings in some particular Banach spaces was studied by many authors e.g., [20,22,27,29,30]; see also books [12,18,31] and some references therein for more details.

In this paper, we deal with the mappings of type (γ). Assume that C is a locally weakly compact convex subset of a Banach space E (i.e. every bounded closed convex subset of C is weakly compact) and $\varphi = \{T_i\}_{i\in I}$ is a family of mappings on C of type (γ) such that $F(\varphi) \neq \varnothing$. From our results in Sections 4 and 5, it can be derived that if either $\cup_{i\in I} F(T_i)$ is contained in a 3-dimensional subspace of E, or $(F(\varphi))^\circ \neq \varnothing$ then $F(\varphi)$ is a nonexpansive retract of C. If, in addition, $F(\varphi)$ is not a line segment then for each $i \in I$ there exists a retraction R_i of type (γ) from C onto $F(\varphi)$, such that $R_iT_i = T_iR_i = R_i$, and every closed convex φ-invariant subset of C is also R_i-invariant. Further, if $F(\varphi)$ is bounded, then there exists a retraction R of type (γ) from $\tilde{C} = \{(x_i) \in l^\infty(E) : x_i \in C, \forall i \in I\}$ onto $F(\varphi) \subset l^\infty(E)$ such that $Ro(T_i) = (T_i)oR = R$ and every closed convex φ-invariant subset of C is R-invariant in $l^\infty(E)$. In Section 6, we prove that if φ is commutative, there exists a retraction R of type (γ) from C onto $F(\varphi)$ such that $RT_i = T_iR = R(\forall i)$, and every closed convex φ-invariant subset of C is R-invariant; the same result holds if φ is a non-commutative right amenable semigroup, under some additional assumptions. In Section 7, we consider affine mappings and under assumptions like those in Section 6, we prove that T_μ (see [26]) is an affine nonexpansive retraction form C onto $F(\varphi)$, such that $T_\mu T_t = T_tT_\mu = T_\mu$ for each $t \in S$, and $T_\mu x \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$. Moreover, if R is an arbitrary retraction from C onto F such that $Rx \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$, then $R = T_\mu$. However our mappings are assumed to be of type (γ), but, compared to the similar works, these results have the merit of studying the existence of retraction for some families of mappings in a general Banach space. We obtain retractions that commute with the mappings (Bruck's results does not assure commutativity). In these theorems, we need not assume the strict convexity of the Banach space. Our results seem to be new even for nonexpansive affine mappings.

2. Preliminaries

Let E be a real normed space and let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for each $x, y \in C$. Let E^* be the topological dual of E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$ or $x^*(x)$. With each $x \in E$, we associate the set

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x^*\|^2 = \|x\|^2\}.$$

Using the Hahn–Banach theorem, it is immediately clear that $Jx \neq \varnothing$ for each $x \in E$. For a subset A of E, we denote by $\overline{co}\, A$ the closed convex hull of A. We denote by Γ the set of strictly increasing, continuous convex functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. For each $\gamma \in \Gamma$, a mapping $T : C \to C$ is said to be of type (γ), if for every $x, y \in C$ and $\lambda \in [0, 1]$,

$$\gamma\left(\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\|\right) \le \|x - y\| - \|Tx - Ty\|.$$

Obviously, if T is of type (γ) for some $\gamma \in \Gamma$, then T is nonexpansive. Furthermore, if C is weakly compact, then $F(T)$ is a nonempty closed convex set (see [12]). If either C is a bounded closed convex subset of a uniformly convex Banach space E, or C is compact and E is a strictly convex Banach space, then there exists $\gamma \in \Gamma$ such that each nonexpansive mapping $T : C \to C$ is of type (γ) (see [5,8]). For more information about mappings of type (γ), see [8,16,25,28].

Throughout this paper, S is a semigroup and $B(S)$ is the space of all bounded real-valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define elements l_sf and r_sf in $B(S)$ by $(l_sf)(t) = f(st)$ and $(r_sf)(t) = f(ts)$ for each $t \in S$, respectively. Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be l_s-and r_s-invariant, i.e. $l_s(X) \subset X$ and $r_s(X) \subset X$ for each $s \in S$. A mean μ on X is said to be right (resp. left) invariant if $\mu(r_sf) = \mu(f)$ (resp. $\mu(l_sf) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be invariant if it is both left and right invariant. X is said to be (right) amenable if there is an (right) invariant mean on X. As is well known, $B(S)$ is amenable when S is a commutative semigroup [10]. A family $\varphi = \{T_s : s \in S\}$ of mappings from C into itself is said to be a semigroup on C if $T_{ts} = T_tT_s$ for each $t, s \in S$. We denote by $F(\varphi)$ the set of common fixed points of φ.

We will use the following definition and notations. By a plane in E we mean a subset of E as $\{\alpha_1z_1 + \alpha_2z_2 + \alpha_3z_3 : \sum\alpha_i = 1, \alpha_i \in \mathbb{R}\}$ where z_1, z_2, z_3 are such points in E that are not on a line; and by a triangle containing z_1, z_2, z_3 we mean the set $\{\alpha_1z_1 + \alpha_2z_2 + \alpha_3z_3 : \sum\alpha_i = 1, \alpha_i \ge 0\}$. Consider $a, b \in E$. By a line segment through a and b we mean one of the following subsets of E:

$$\langle a, b\rangle = \{\alpha a + (1-\alpha)b : \alpha \in \mathbb{R}\};$$
$$(a, b) = \{\alpha a + (1-\alpha)b : 0 < \alpha < 1\};$$
$$[a, b] = \{\alpha a + (1-\alpha)b : 0 \le \alpha \le 1\};$$