Nonlinear Analysis 71 (2009) 1235–1249

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Bifurcation of sign-changing solutions for one-dimensional p-Laplacian with a strong singular weight; p-sublinear at ∞

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ARTICLE INFO

Article history:
Received 30 July 2008
Accepted 6 November 2008

MSC:
34B09
34B16
34C23

Keywords:
p-Laplace equation
Singular weight
Leray–Schauder degree
Bifurcation
Positive solutions
Sign-changing solutions

ABSTRACT

In this paper, we consider one-dimensional p-Laplacian problems with a singular weight which may not be in L^1. Using the properties of eigenfunctions and the global bifurcation theory, we prove existence, uniqueness, nonexistence and multiplicity results of positive solutions as well as sign-changing solutions specially when the nonlinear term is p-linear near 0 and p-sublinear at ∞.

1. Introduction

In this paper, we consider the following one-dimensional p-Laplacian problem with a coefficient function

$$\begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0, & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{P$_\lambda$}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, λ is a nonnegative parameter, $f \in C(\mathbb{R}, \mathbb{R})$ and h is a nonnegative measurable function on $(0, 1)$, $h \not\equiv 0$ on any compact subinterval in $(0, 1)$.

Existence, nonexistence and multiplicity of nontrivial solutions for problems of the form (P_λ) were recently studied by several authors, e.g., for positive solutions by [1–7] and for sign-changing solutions by [8–11].

Our main concern here is to study the existence of positive solutions as well as sign-changing solutions when f satisfies

(F1) $sf(s) > 0$ for $s \neq 0$,
(F2) $0 < f_0 \equiv \lim_{s\to 0} f(s)/\varphi_p(s) < \infty$,
(F3) $f_\infty \equiv \lim_{|s|\to\infty} f(s)/\varphi_p(s) = 0$

and specially when the weight h is singular at the boundary with possibility $h \notin L^1(0, 1)$.

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doi:10.1016/j.na.2008.11.056

For the continuous weight case, Naito–Tanaka [11,12] recently studied the existence of positive solutions and sign-changing solutions. Assuming (F1), (F2), (F3) and

(NT1) f is locally Lipschitz continuous on $\mathbb{R} \setminus \{0\}$
(NT2) $h \in C^1[0, 1]$ and $h > 0$, on $[0, 1]$,

they prove that for each $k \in \mathbb{N}$, problem (P_λ) has at least two $(k-1)$-nodal solutions for $\lambda \in (\mu_k, \infty)$ and no $(k-1)$-nodal solution for $\lambda \in (0, \frac{f_0}{f^*}\mu_k)$, where μ_k is the kth eigenvalue of

$$\begin{cases} \varphi_p(u'(t))' + \lambda f_0 h(t)\varphi_p(u(t)) = 0, & \text{a.e. in } (0, 1), \\ u(0) = u(1) = 0 \end{cases} \tag{E_λ}$$

and $f^* \triangleq \sup_{s\in\mathbb{R}\setminus\{0\}} \frac{f(s)}{\varphi_p(s)}$. Here, we mean (λ, u) a *k-nodal solution* of (P_λ). if u has exactly k zeros in the interval $(0, 1)$. Thus a 0-nodal solution means a positive solution or a negative solution.

Proofs are mainly employed by the shooting method together with the qualitative theory for half-linear differential equations. One may refer to Lee–Sim [8] who give a partial result for the existence of sign-changing solutions when $h \in L^1(0, 1)$. Precisely, assuming (F1),(F2) and (F3) with oddity of f, they prove that for each $k \in \mathbb{N}$, problem (P_λ) has at least two $(k-1)$-nodal solutions for $\lambda \in (\mu_k, \infty)$. One also refers to Zhang [13] for the existence of eigenvalues and properties like (i)–(iv) in Theorem 3.1 of (E_λ) up to $h \in L^1((0, 1), \mathbb{R}_+)$ where $\mathbb{R}_+ = [0, \infty)$.

For the singular weight case, Agarwal–Lü–O'Regan [1] studied the existence of positive solutions. Assume (F2) and (F3) for $f \in C(\mathbb{R}_+, \mathbb{R}_+)$. Also assume that h satisfies

$$\int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s h(\tau)\mathrm{d}\tau\right)\mathrm{d}s < \infty.$$

Then they prove that (P_λ) has at least one positive solution for $\lambda \in (\frac{1}{f_0\varphi_p(\alpha)}, \infty)$, where α is determined by the above double integral condition on h.

In what follows, it is interesting to consider the existence of sign-changing solutions under the above double integral condition. But as far as the authors know, this is highly not obvious by the following two reasons;

(I) Well definedness of corresponding operator equation for sign-changing solutions is not clear.
(II) Isolation of eigenvalues of (E_λ) is not guaranteed. (The authors understand that this is a difficult question, see [14]).

As the main condition of h for this paper, we introduce the following;

$$\int_0^1 s^{p-1}(1-s)^{p-1}h(s)\mathrm{d}s < \infty.$$

Recently, the authors [15] prove the existence of eigenvalues of (E_λ) when h satisfies the integral condition (see Theorem 3.1 in Section 3). Based on this theorem, we will apply bifurcation theory to study the existence of sign-changing solutions for (P_λ). Our results extend those of Naito–Tanaka [11], Lee–Sim [8] as well as Agarwal–Lü–O'Regan [1] in some directions.

There are some interesting relations between the above two integral conditions. For convenience, let

$$\mathcal{A} \equiv \left\{h \in L^1_{loc}(0, 1) : \int_0^1 s^{p-1}(1-s)^{p-1}h(s)\mathrm{d}s < \infty\right\}.$$

$$\mathcal{B} \equiv \left\{h \in L^1_{loc} : \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s h(\tau)\mathrm{d}\tau\right)\mathrm{d}s < \infty\right\}.$$

Then $L^1(0, 1) \subset \mathcal{A} \cap \mathcal{B}$ and classes $\mathcal{A}$ and $\mathcal{B}$ are equivalent when $p = 2$. It is also known [16] that $\mathcal{A} \subsetneqq \mathcal{B}$ for $1 < p < 2$ and $\mathcal{B} \subsetneqq \mathcal{A}$ for $p > 2$.

C^1-regularity of solution space for (P_λ) is not obvious mainly due to the singular effect of h. But Proposition 2.1 in Section 2 guarantees to find C^1 solutions for the problem. Denote by $\mathcal{S}$ the closure of the set of nontrivial solutions and we state one of our main theorems in this paper.

Theorem 1.1. *Assume that $h \in \mathcal{A}$. Also assume that* (F1) *and* (F2) *. Then for each $k \in \mathbb{N}$, there exist two unbounded subcontinua $\mathcal{C}_k^\pm$ in $\mathcal{S}$ bifurcating from $(\mu_k, 0)$. Furthermore, $\mathcal{C}_k^\pm \cap (\mathbb{R} \times \{0\}) = \{(\mu_k, 0)\}$ and if $(\lambda, u) \in \mathcal{C}_k^+ \setminus \{(\mu_k, 0)\}(\mathcal{C}_k^- \setminus \{(\mu_k, 0)\})$, then u is a $(k-1)$-nodal solution in $(0, 1)$ satisfying $u'(0) > 0(u'(0) < 0)$, respectively.*

Based on this theorem and under assumptions $h \in \mathcal{A}$, (F1),(F2) and (F3), we prove that for each $k \in \mathbb{N}$, (P_λ) has at least two $(k-1)$-nodal solutions (one in $\mathcal{C}_k^+$ and the other in $\mathcal{C}_k^-$) for $\lambda \in (\mu_k, \infty)$ and no $(k-1)$-nodal solutions for $\lambda \in (0, \frac{f_0}{f^*}\mu_k)$. See Theorem 5.1 in Section 5. Also investigating the figure of bifurcation branches $\mathcal{C}_k^\pm$ with additional conditions on f, we obtain several results about existence, uniqueness, nonexistence and multiplicity of positive solutions and sign-changing solutions. See Theorems and Corollary 5.2–5.6.