Available online at www.sciencedirect.com

ScienceDirect

Nonlinear Analysis 69 (2008) 1494–1514

Nonlinear Analysis

www.elsevier.com/locate/na

A multi-dimensional quenching problem due to a concentrated nonlinear source in $\mathbb{R}^N$

C.Y. Chan*, P. Tragoonsirisak

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA

Received 9 May 2007; accepted 6 July 2007

Abstract

Let B be a N-dimensional ball $\{x \in \mathbb{R}^N : |x| < R\}$ centered at the origin with a radius R, $\bar{B}$ be its closure, and ∂B be its boundary. Also, let $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and $\chi_B(x)$ be the characteristic function, which is 1 for $x \in B$, and 0 for $x \in \mathbb{R}^N \backslash B$. This article studies the following multi-dimensional semilinear parabolic first initial-boundary value problem with a concentrated nonlinear source on ∂B:

$$u_t - \triangle u = \alpha \frac{\partial \chi_B(x)}{\partial \nu} f(u) \quad \text{in } \mathbb{R}^N \times (0, T],$$
$$u(x, 0) = 0 \quad \text{for } x \in \mathbb{R}^N, \qquad u(x, t) \to 0 \quad \text{as } |x| \to \infty \text{ for } 0 < t \leq T,$$

where α and T are positive numbers, f is a given function such that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \leq u < c$. It is shown that the problem has a unique nonnegative continuous solution before quenching occurs, and if u quenches in a finite time, then it quenches everywhere on ∂B only. It is proved that u always quenches in a finite time for $N \leq 2$. For $N \geq 3$, it is shown that there exists a unique number α^* such that u exists globally for $\alpha \leq \alpha^*$ and quenches in a finite time for $\alpha > \alpha^*$. Thus, quenching does not occur in infinite time. A formula for computing α^* is given. A computational method for finding the quenching time is devised.

MSC: primary 35K60; 35K57; 35B35

Keywords: Multi-dimensional quenching; Concentrated nonlinear source; Unbounded domain; Existence; Uniqueness; Quenching; Computational method; Quenching time

1. Introduction

Let $H = \partial/\partial t - \triangle$, T be a positive real number, $x = (x_1, x_2, \ldots, x_N)$ be a point in the N-dimensional Euclidian space $\mathbb{R}^N$, $\Omega = \mathbb{R}^N \times (0, T]$, B be a N-dimensional ball $\{x \in \mathbb{R}^N : \left|x - \bar{b}\right| < R\}$ centered at a given point $\bar{b}$ with a radius R, ∂B be the boundary of B, $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and

$$\chi_B(x) = \begin{cases} 1 & \text{for } x \in B, \\ 0 & \text{for } x \in \mathbb{R}^N \backslash B, \end{cases}$$

* Corresponding author. Tel.: +1 337 482 5288; fax: +1 337 482 5346.
E-mail addresses: chan@louisiana.edu (C.Y. Chan), pxt6365@louisiana.edu (P. Tragoonsirisak).

0362-546X/$ - see front matter
doi:10.1016/j.na.2007.07.001

be the characteristic function. We would like to study the following multi-dimensional semilinear parabolic first initial-boundary value problem with a source on the surface of the ball:

$$\left.\begin{aligned} &Hu = \alpha \frac{\partial \chi_B(x)}{\partial \nu} f(u) \quad \text{in } \Omega, \\ &u(x,0) = 0 \text{ for } x \in \mathbb{R}^N, \qquad u(x,t) \to 0 \quad \text{as } |x| \to \infty \text{ for } 0 < t \le T, \end{aligned}\right\} \tag{1.1}$$

where α is a positive number. Without loss of generality, let $\bar{b}$ be the origin. This model is motivated by a N-dimensional ball B having a radius R and situated in $\mathbb{R}^N$; on the surface ∂B of the ball, there is a nonlinear heat source of strength $\alpha f(u)$, where $u(x,t)$ in Ω is the unknown temperature to be determined. We assume that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \le u < c$. A solution u is said to quench if there exists an extended real number $t_q \in (0, \infty]$ such that

$$\sup \left\{ u(x,t) : x \in \mathbb{R}^N \right\} \to c^- \quad \text{as } t \to t_q.$$

If $t_q < \infty$, then u is said to quench in a finite time. If $t_q = \infty$, then u quenches in infinite time. We note that whether quenching for the heat equation with a non-concentrated source in an unbounded domain occurs in a finite time was studied by Dai and Gu [1], and Dai and Zeng [2].

In Section 2, we show that the nonlinear integral equation corresponding to the problem (1.1) has a unique nonnegative continuous solution u, which is a strictly increasing function of t. We then prove that u is the unique solution of the problem (1.1). If t_q is finite, we show that u quenches everywhere on ∂B only. In Section 3, we prove that for $N \le 2$, u always quenches in a finite time. This behavior is completely different from that for $N \ge 3$. In Section 4, we show that for $N \ge 3$, there exists a unique number α^* such that u exists globally for $\alpha \le \alpha^*$ and quenches in a finite time for $\alpha > \alpha^*$. This rules out the possibility of quenching in infinite time. We also derive a formula for computing α^*. In Section 5, we give a computational method for determining the finite quenching time t_q.

2. Existence, uniqueness, and quenching

Green's function $g(x,t;\xi,\tau)$ (cf. Stakgold [3, p. 198]) corresponding to the problem (1.1) is determined by the following system:

$$\begin{aligned} &Hg(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau) \quad \text{for } x, \xi \in \mathbb{R}^N \text{ and } t, \tau \in (-\infty, \infty), \\ &g(x,t;\xi,\tau) = 0 \quad \text{for } x, \xi \in \mathbb{R}^N, \text{ and } t < \tau, \\ &g(x,t;\xi,\tau) \to 0 \quad \text{as } |x| \to \infty \text{ for } \xi \in \mathbb{R}^N \text{ and } t, \tau \in (-\infty, \infty). \end{aligned}$$

It is given by

$$g(x,t;\xi,\tau) = \begin{cases} \dfrac{1}{[4\pi(t-\tau)]^{N/2}} \exp\left(-\dfrac{|x-\xi|^2}{4(t-\tau)}\right), & t > \tau, \\ 0, \quad t < \tau. \end{cases}$$

To derive the integral equation from the problem (1.1), let us consider the adjoint operator $(-\partial/\partial t - \triangle)$ of H. Using Green's second identity, we obtain

$$u(x,t) = \alpha \int_0^t \int_{\mathbb{R}^N} g(x,t;\xi,\tau) \frac{\partial \chi_B(\xi)}{\partial \nu} f(u(\xi,\tau)) \mathrm{d}\xi \mathrm{d}\tau.$$

Let $\tilde{R}$ denote a positive number larger than R. Then,

$$\int_{\mathbb{R}^N} g(x,t;\xi,\tau) \frac{\partial \chi_B(\xi)}{\partial \nu} f(u(\xi,\tau)) \mathrm{d}\xi = \lim_{\tilde{R} \to \infty} \int_{|\xi| < \tilde{R}} g(x,t;\xi,\tau) f(u(\xi,\tau)) \left[\nu(\xi) \cdot \nabla \chi_B(\xi)\right] \mathrm{d}\xi.$$

Since $\chi_B(\xi) = 0$ for $\xi \in \mathbb{R}^N \backslash B$, it follows from an integration by parts that

$$\lim_{\tilde{R} \to \infty} \int_{|\xi| < \tilde{R}} g(x,t;\xi,\tau) f(u(\xi,\tau)) \left[\nu(\xi) \cdot \nabla \chi_B(\xi)\right] \mathrm{d}\xi$$