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Linearized stability for a class of neutral functional differential equations with state-dependent delays

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Abstract

In this paper we formulate a stability theorem by means of linearization around a trivial solution in the case of autonomous neutral functional differential equations with state-dependent delays. We prove that if the trivial solution of the linearized equation is exponentially stable, then the trivial solution of the nonlinear equation is exponentially stable as well. As an application of the main result, explicit stability conditions are given.

MSC: 34K20; 34K40

Keywords: Neutral equation; State-dependent delay; Linearization; Exponential stability

1. Introduction and formulation of the main results

In this paper, we consider the autonomous neutral differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - g(x(t-\sigma(x_t)))\right) = f\left(x_t, x(t-\tau(x_t))\right), \quad t \geq 0 \tag{1.1}$$

and the associated initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0], \ \varphi \in C. \tag{1.2}$$

Here, we assume that $r > 0$ is fixed, $g\colon \mathbb{R}^n \to \mathbb{R}^n$, $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma, \tau\colon C \to [0, r]$. A fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$. C is the Banach space of continuous functions $\psi\colon [-r, 0] \to \mathbb{R}^n$ equipped with the norm $\|\psi\| = \sup\{|\psi(s)|\colon s \in [-r, 0]\}$. The solution segment function $x_t\colon [-r, 0] \to \mathbb{R}^n$ is defined by $x_t(s) = x(t+s)$.

We assume that $x = 0$ is a constant equilibrium of (1.1), and we study the exponential stability of the trivial solution by means of the linearization technique.

For retarded delay differential equations with state-dependent delays (SD-DDEs), i.e., in the case when $g \equiv 0$ in (1.1), a linearized stability theorem was first proved in [5]. Later, similar results were proved for different classes

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of SD-DDEs in [12,16,20,21,27]. The main technical difficulty to prove a linearized stability theorem in SD-DDEs is that the map $C \ni \psi \mapsto f(\psi, \psi(-\tau(\psi))) \in \mathbb{R}^n$ is not Fréchet-differentiable. See [22,27] for more details and discussions on this topic. We refer the interested reader also to [22] for a survey on the general theory and applications of SD-DDEs. The study of SD-DDEs is an active research area (see, e.g., [1,9,16,22,24,25] and the references therein). Much less work is devoted to neutral functional differential equations with state-dependent delays [2–4,6,11,17–19, 23,28,29].

We compare the exponential stability of the trivial solution of (1.1) to that of the associated linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(y(t) - g'(0)y(t-\sigma(\mathbf{0}))\big) = D_1 f(\mathbf{0}, 0)y_t + D_2 f(\mathbf{0}, 0)y(t-\tau(\mathbf{0})), \quad t \ge 0, \tag{1.3}$$

where $\mathbf{0}$ is the constant 0 function in C, and we associate initial condition (1.2) to (1.3).

We assume throughout the paper

(H1) (i) the function $g: U_1 \to \mathbb{R}^n$ is continuously differentiable, where $U_1 \subset \mathbb{R}^n$ is open, and $0 \in U_1$;
(ii) $g(0) = 0$;
(iii) $|g'(0)| < 1$;

(H2) (i) the function $f: U_2 \times U_3 \to \mathbb{R}^n$ is continuously differentiable, where $U_2 \subset C$ and $U_3 \subset \mathbb{R}^n$ are open subsets, $\mathbf{0} \in U_2$ and $0 \in U_3$;
(ii) $f(\mathbf{0}, 0) = 0$;

(H3) (i) the delay functions $\sigma, \tau: U_4 \to [0, r]$ are continuous, where $U_4 \subset C$ is open, and $\mathbf{0} \in U_4$;
(ii) $\sigma(\mathbf{0}) \ne 0$;

(H4) $\varphi \in C$.

Note that (H1)(ii) is not a restriction on the problem, since we can always add a constant to the function g.

Assumptions (H1)–(H4) yield only the existence, but not the uniqueness of solutions of the IVP (1.1), (1.2) (see corresponding results for retarded SD-DDEs, e.g., in [7,20,22]).

We say that the trivial (zero) solution of the linear equation (1.3) is exponentially stable if there exists $K_1 \ge 0$ and $\alpha > 0$ such that

$$|y(t)| \le K_1 \mathrm{e}^{-\alpha t}\|\varphi\|, \quad t \ge 0. \tag{1.4}$$

In this case, we say that the order of exponential stability is α.

Similarly, we say the trivial solution of the nonlinear equation (1.1) is exponentially stable if there exist $K \ge 0$, $\theta > 0$ and $\delta > 0$ such that

$$|x(t)| \le K\mathrm{e}^{-\theta t}\|\varphi\|, \quad t \ge 0, \ \|\varphi\| \le \delta.$$

We formulate the main result of the paper in the next theorem.

Theorem 1.1. *Assume* (H1)–(H4). *If the trivial solution of* (1.3) *is exponentially stable, then the trivial solution of* (1.1) *is exponentially stable as well.*

The proof will be given in two steps. In Section 3, we show that the trivial solution of (1.1) is stable, and in Section 4 we give the proof for its exponential stability. Section 2 contains some preliminary results and introduces notations that will be used in the sequel.

We comment that the results are only presented here for the case of the zero equilibrium, but they are easy to generalize for any constant equilibrium. Also, the proofs are easy to extend to cases when there are multiple state-dependent delay terms on the right-hand-side of (1.1), but the method which we use (especially Proposition 2.3) relies on the fact that there is only a single delay term in the neutral part of the equation, i.e., on the left-hand-side of (1.1).

Theorem 1.1 immediately has the following corollary. Let I be the $n \times n$ identity matrix.

Corollary 1.2. *Assume* (H1)–(H4). *If there exists a* $c_0 > 0$ *such that all roots of*

$$\lambda I - g'(0)\lambda \mathrm{e}^{-\lambda\sigma(\mathbf{0})} = D_1 f(\mathbf{0}, 0)\left(\mathrm{e}^{\lambda\cdot} I\right) + D_2 f(\mathbf{0}, 0)\mathrm{e}^{-\lambda\tau(\mathbf{0})}$$

satisfy $\mathrm{Re}\,\lambda \le -c_0$, *then trivial solution of* (1.1) *is exponentially stable.*