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A general iterative method for an infinite family of nonexpansive mappings

Yonghong Yao[a,*], Yeong-Cheng Liou[b], Rudong Chen[a]

[a] Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
[b] Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan

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Abstract

Let H be a real Hilbert space. Consider the iterative sequence

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n x_n,$$

where $\gamma > 0$ is some constant, $f : H \to H$ is a given contractive mapping, A is a strongly positive bounded linear operator on H and W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1, T_2, \ldots, T_n, \ldots$ and $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ such that the common fixed points set $F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. Under very mild conditions on the parameters, we prove that $\{x_n\}$ converges strongly to $p \in F$ where p is the unique solution in F of the following variational inequality:

$$\langle (A - \gamma f)p, p - x^* \rangle \leq 0 \quad \text{for all } x^* \in F,$$

which is the optimality condition for the minimization problem

$$\min_{x \in F} \frac{1}{2} \langle Ax, x \rangle - h(x).$$

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Keywords: Nonexpansive mapping; Fixed point; Minimization problem; Variational inequality; Iterative method

1. Introduction

Let H be a real Hilbert space, let A be a bounded linear operator on H. In this paper, we always assume that A is *strongly positive*; that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \forall x \in H$. Recall that a mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. Recall also that a *contraction* on H is a self-mapping f of H such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|, \forall x, y \in H$, where $\alpha \in [0, 1)$ is a constant.

* Corresponding author.
E-mail addresses: yuyanrong@tjpu.edu.cn (Y. Yao), simplex_liou@hotmail.com (Y.-C. Liou), chenrd@tjpu.edu.cn (R. Chen).

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Finding an optimal point in the intersection F of the fixed points set of a family of nonexpansive mappings is one that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed points set of a family of nonexpansive mappings; see, e.g., [1,2]. The problem of finding an optimal point that minimizes a given cost function $\Theta : H \to \mathbf{R}$ over F is of wide interdisciplinary interest and practical importance, see, e.g., [3–6]. A simple algorithmic solution to the problem of minimizing a quadratic function over F is of extreme value in many applications including the set theoretic signal estimation, see, e.g., [6,7]. The best approximation problem of finding the projection $P_F(a)$ (in the norm induced by the inner product of H) from any given point a in H is the simplest case of our problem.

Let T_i, where $i = 1, 2, \dots, N$, a finite family of nonexpansive self-mappings of H. Let $\operatorname{Fix}(T_i)$ denote the fixed points set of T_i, i.e.,

$$\operatorname{Fix}(T_i) := \{x \in H : T_i x = x\},$$

and let $F := \bigcap_{i=1}^N \operatorname{Fix}(T_i)$. We are concerned with the following *quadratic minimization problem*:

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1}$$

where $b \in H$ and $A : H \to H$ is a self-adjoint strongly positive operator.

Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \alpha_n A)T_n x_n + \alpha_n u, \quad n \geq 0, \tag{2}$$

where $T_n = T_{\text{n mod N}}$ and the mod function takes values in $\{1, 2, \dots, N\}$. It is proved [8] that the sequence $\{x_n\}$ generated by (2) converges strongly to the unique solution of the minimization problem (1) provided $\{T_n\}$ satisfy

$$F = \operatorname{Fix}(T_N \cdots T_2 T_1) = \operatorname{Fix}(T_1 T_N \cdots T_3 T_2) = \cdots = \operatorname{Fix}(T_{N-1} \cdots T_1 T_N), \tag{3}$$

and $\{\alpha_n\}$ is a sequence in (0, 1) satisfying the following control conditions

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^\infty \alpha_n = \infty$;
(C3) $\sum_{n=1}^\infty |\alpha_n - \alpha_{n+N}| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+N}} = 1$.

Remark 1.1. (1) The convergence problem of the iterative scheme (2) for one, finitely or infinitely-many nonexpansive mappings $T_1, T_2, \dots$ in the settings of Hilbert spaces or some special Banach spaces has been introduced and studied by many authors. See, for example [3,9–13] and the references therein. We note that all of the above results have imposed some additional assumptions on parameters $\{\alpha_n\}$ or mappings $\{T_n\}$. For the details, please see [3,9–13].

(2) We observe that there are many nonexpansive mappings that do not satisfy (3). For example, if $X = [0, 1]$ and T_1 and T_2 are defined by $T_1 x = \frac{x}{2} + \frac{1}{4}$ and $T_2 x = \frac{3x}{4}$, then $\operatorname{Fix}(T_1 T_2) = \{\frac{2}{5}\}$, whereas $\operatorname{Fix}(T_2 T_1) = \{\frac{3}{10}\}$.

Very recently, Marino and Xu [14] considered a general iterative method for a single nonexpansive mapping. Let f be a contraction on H, $A : H \to H$ be a strongly positive bounded linear operator. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T x_n, \quad n \geq 0, \tag{4}$$

where $\gamma > 0$ is a constant and $\{\alpha_n\}$ is a sequence in (0, 1) satisfying conditions (C1), (C2) and (C3) with $N = 1$. Consequently, Marino and Xu [14] proved that the sequence $\{x_n\}$ generated by (4) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in \operatorname{Fix}(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in \operatorname{Fix}(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).