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The convexity of the solution set of a pseudoconvex inequality

Zili Wu*

Ninth Middle School (Jingu), Anxi County, Fujian 362400, PR China

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Abstract

For a pseudoconvex function f on a nonempty convex set C in a real normed vector space X, we present several equivalent conditions for the convexity of the set

$$C_x := \{c \in C : f(x) \leq f(c)\} \quad \text{for } x \in C.$$

These conditions turn out to be very useful in characterizing the solution set of a pseudoconvex minimization problem of f over C_x and the pseudolinearity of a Gâteux differentiable function f. We hence extend several existing results about characterizations of the solutions to a convex program and a pseudolinear program.

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1. Introduction

Let C be a nonempty convex set in a real normed vector space X whose dual space is X^*. We say that a function $f : X \to (-\infty, +\infty)$ is *pseudoconvex* on C if it is Gâteaux differentiable at each $x \in C$ and, for $y \in C$,

$$\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x),$$

where $\nabla f(x)$ is the Gâteaux derivative of f at x which lies in X^* and satisfies

$$\langle \nabla f(x), v \rangle = \lim_{t \to 0^+} \frac{f(x + tv) - f(x)}{t} \quad \text{for all } v \in X.$$

If $-f$ is pseudoconvex on C, then f is said to be *pseudoconcave* on C. A function f is *pseudolinear* on C if it is both pseudoconvex and pseudoconcave on C. The concept of pseudoconvexity was introduced by Mangasarian in [9] to generalize that of the convexity of a differentiable function.

For $(x, y) \in C \times C$, we denote

$$xty := x + t(y - x) \quad \text{for } t \in [0, 1];$$

* Tel.: +1 250 382 0026.
E-mail address: ziliwu@email.com.

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$[x, y] := \{xty : t \in [0, 1]\}$;
$C_x := \{c \in C : f(x) \le f(c)\}$;
$D_x := \{c \in C : f(x) \ge f(c)\}$;
$E_x := \{c \in C : f(x) = f(c)\}$.

Clearly E_x is the solution set of the programming problem to minimize f over C_x and

$$f(x) = f(y) \Leftrightarrow C_x = C_y \Leftrightarrow D_x = D_y \Leftrightarrow E_x = E_y.$$

In addition, for a pseudoconvex function f on C,

$$C = \{c \in C : \langle \nabla f(x), c - x\rangle \ge 0\} \Leftrightarrow C_x = C.$$

When $C_x = C$, the solution set of the minimization problem of f over C is

$$E_x = \{y \in C : \langle \nabla f(y), c - y\rangle \ge 0 \text{ for all } c \in C\}.$$

It is interesting to characterize the solution set of a minimization problem so that we can know more about the nature of its solutions. In R^n, when x is an optimal solution, the set E_x has been characterized in terms of the gradient ∇f by Mangasarian in [10] for a twice continuously differentiable convex program and by Jeyakumar and Yang [6] for a pseudolinear program. For the nondifferentiable convex case, the reader is referred to [2,5,10,12].

We note that the convexity of C_x plays an important role in characterizing the set E_x (no matter whether $C_x = C$ or not). Our main purpose in this paper is to present several equivalent conditions for the convexity of C_x in the pseudoconvex case. With these conditions we also characterize E_x to unify the corresponding results in [10,6]. Furthermore we use them to derive several results about the pseudolinearity of a function in a normed vector space under weaker conditions than in [3,7,8,11].

2. The convexity of C_x for a pseudoconvex function

It is easy to see that C_x may be nonconvex for each $x \in C$ with $C_x \ne C$ even for the convex case. For example, when $f(x) = x^2$ for $x \in C = [-1, 2]$, C_x is not convex for each $x \in [-1, 0) \cup (0, 1]$. However, for $x \in (1, 2]$, C_x is convex. We note that $\langle \nabla f(x), y - x\rangle \ge 0$ for all $x \in (1, 2]$ and for all $y \in C_x$. Indeed, this is one of the equivalent conditions for C_x to be convex as we state in our first simple result below.

Proposition 2.1. *Let f be pseudoconvex at $x \in C$. Then the following are equivalent:*

(i) *C_x is convex.*
(ii) *For each $y \in C_x$, $[x, y] \subseteq C_x$.*
(iii) *$\langle \nabla f(x), y - x\rangle \ge 0$ for all $y \in C_x$.*

Proof. It suffices to show (iii) ⇒ (i) since (i) ⇒ (ii) ⇒ (iii) is trivial.

Let (iii) be true. Then for any $y_1, y_2 \in C_x$,

$$\langle \nabla f(x), y_1 - x\rangle \ge 0 \quad \text{and} \quad \langle \nabla f(x), y_2 - x\rangle \ge 0,$$

so $\langle \nabla f(x), y_1ty_2 - x\rangle \ge 0$ for all $t \in [0, 1]$. By the pseudoconvexity of f at x, $y_1ty_2 \in C_x$ for all $t \in [0, 1]$. This shows (i). ■

In Proposition 2.1, we only assume that the function f is pseudoconvex at the point x. When the function is pseudoconvex on the whole set C_x we have more equivalent statements for the convexity of C_x.

Theorem 2.2. *Let C be a nonempty convex set in a normed vector space X and $x \in C$. If f is pseudoconvex on C_x, then* (i)–(iii) *in Proposition 2.1 and the following are equivalent:*

(iv) *For each $y \in C_x$ there exist $p(x, y) > 0$ and $q(x, y) > 0$ such that $p(x, y) \le 1$ and*

$$f(y) \ge f(x) + p(x, y)\langle \nabla f(x), y - x\rangle \ge f(x) + q(x, y)\langle \nabla f(x), y - x\rangle^2.$$