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Basic theory of p-Amemiya norm in Orlicz spaces ($1 \leq p \leq \infty$): Extreme points and rotundity in Orlicz spaces endowed with these norms

Yunan Cui[a], Lifen Duan[b], Henryk Hudzik[c,*], Marek Wisła[c]

[a] *Department of Mathematics, Harbin University of Science and Technology, Harbin, 150080, PR China*
[b] *Department of Mathematics, TongHua Normal College, TongHua, 134002, PR China*
[c] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland*

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Abstract

In any modular space generated by a convex modular we define a family of new norms (called p-Amemiya norms) which are equivalent to the Orlicz norm as well as to the Luxemburg norm. Next, the new case of Orlicz spaces is studied carefully. The attainable points of the p-Amemiya norm in Orlicz function spaces generated by N-functions are discussed. The intervals for p-Amemiya norm attainability are described. Criteria for extreme points as well as for rotundity of Orlicz function spaces endowed with p-Amemiya norm are given. The obtained results unify, complete and extend as well the results presented by a number of papers devoted to studying the geometry of Orlicz spaces endowed with the Luxemburg norm and the Orlicz norm separately. From our results it follows that there are Orlicz spaces which are rotund for p-Amemiya norm with $1 < p < \infty$ only, that is, they are neither rotund for the Luxemburg norm corresponding to the case $p = \infty$ nor for Orlicz norm corresponding to the case $p = 1$.

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1. Introduction

Let X be a real vector space. A function $\rho : X \to [0, \infty)$ is called a convex modular if it satisfies the following conditions:

(1) $\rho(0) = 0$ and $x = 0$ whenever $x \in X$ and $\rho(ax) = 0$ for any $a > 0$;
(2) $\rho(-x) = \rho(x)$;
(3) $\rho(ax + by) \leq a\rho(x) + b\rho(y)$, for $x, y \in X, a, b \geq 0, a + b = 1$

(see [15–18]).

Define

* Corresponding author. Tel.: +48 61 8295356; fax: +48 61 8295315.
E-mail addresses: cuiya@mail.hrbust.edu.cn (Y. Cui), hudzik@amu.edu.pl (H. Hudzik), mwisla@amu.edu.pl (M. Wisła).

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$$X_\rho = \{x \in X : \lim_{\lambda \to 0^+} \rho(\lambda x) = 0\}.$$

For any $x \in X_\rho$, the Luxemburg norm is defined by

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho(\lambda^{-1} x) \le 1\}$$

and the Orlicz norm is defined by the Amemiya formula:

$$\|x\|_\rho^o = \inf_{k>0} k^{-1} (1 + \rho(kx)).$$

Now, for $1 \le p \le \infty$, on X_ρ we introduce a new functional as follows

$$\|x\|_{\rho,p} = \begin{cases} \inf\limits_{k>0} k^{-1} \left(1 + \rho^p(kx)\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \inf\limits_{k>0} k^{-1} \max\{1, \rho(kx)\}, & \text{if } p = \infty. \end{cases}$$

Evidently $\|x\|_{\rho,1} = \|x\|_\rho^o$. In the next section we shall recall that $\|x\|_{\rho,\infty} = \|x\|_\rho$ (see [21]). We will also prove there that, for any $1 \le p \le \infty$, the functional $\|x\|_{\rho,p}$ is a norm on X_ρ and all the norms $\|x\|_{\rho,p}$ are equivalent to each other. We will call the functional $\|x\|_{\rho,p}$ as p-Amemiya norm on X_ρ.

Denote by $\mathcal{R}$ the set of real numbers. A mapping $M : \mathcal{R} \to [0, \infty]$ is called an Orlicz function if M is even, left-continuous on $\mathcal{R}_+ = [0, \infty)$, convex, $M(0) = 0$ and $\lim_{u\to\infty} M(u) = \infty$. We extend the definition of M assuming $M(\infty) = \infty$. Denote $a_M = \sup\{u \ge 0 : M(u) = 0\}$ and $c_M = \sup\{u \ge 0 : M(u) < \infty\}$. Since M is convex and vanishes at 0 we have

$$0 \le \frac{M(u)}{u} \le \frac{M(v)}{v} \quad \text{for every } 0 \le u \le v. \tag{1}$$

For every Orlicz function M, we define its complementary function $N : \mathcal{R} \to [0, \infty)$ by the formula

$$N(v) = \sup_{u\ge 0}(u|v| - M(u)).$$

It is well-known that the complementary function is an Orlicz function as well. Throughout the paper, by M, N we shall always mean a pair of Orlicz functions that are complementary to each other.

Let (G, Σ, μ) be a measure space with a σ-finite, nonatomic and complete measure μ and L^0 be the set of all μ-equivalence classes of real Σ-measurable functions defined on G. We shall say that an Orlicz function M satisfies the Δ_2-condition ($M \in \Delta_2$ in short), if there exists a number $K > 0$ such that $M(2u) \le KM(u)$ for all $u \ge 0$ (respectively, for every $u \ge u_0$ and some $u_0 > 0$ with $M(u_0) < \infty$) if $\mu(G) = \infty$ (resp., $\mu(G) < \infty$).

For any real-valued measurable function x on G, define

$$\rho_M(x) = \int_G M(x(t))\mathrm{d}\mu.$$

It is clear that $\rho_M(x)$ is a convex modular. The modular space

$$X_{\rho_M} = \{x \in L^0 : \lim_{\lambda\to 0^+} \rho_M(\lambda x) = 0\} = \{x \in L^0 : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

is called the Orlicz function space.

Denote by $\|\cdot\|_M$, $\|\cdot\|_M^o$, $\|\cdot\|_{M,p}$ ($1 \le p \le \infty$) the Luxemburg norm, the Orlicz norm and the p-Amemiya norm on the space X_{ρ_M}, respectively. To follow up the standard notations, instead of $\|x\|_{M,1}$ (resp., $\|x\|_{M,\infty}$) we shall write $\|x\|_M^o$ (resp., $\|x\|_M$). To simplify notations, we put $L_M = (X_{\rho_M}, \|\cdot\|_M)$, $L_M^o = (X_{\rho_M}, \|\cdot\|_M^o)$ and $L_{M,p} = (X_{\rho_M}, \|\cdot\|_{M,p})$. Further, for any function $x \in L^0$ the essential supremum of $|x|$ over G, i.e., $\sup \mathrm{ees}_{t\in G}|x(t)|$, no matter whether this number is finite or not, will be denoted by $\|x\|_\infty$.

Let us note, that in [19] Orlicz defined the norm on X_{ρ_M} by use of another formula

$$\|x\|_M^o = \sup\left\{\int_G x(t)y(t)\mathrm{d}\mu(t) : \rho_N(y) \le 1\right\}.$$

However, H. Hudzik and L. Maligranda (see [8]) proved that the above norm coincides with the norm defined by the Amemiya formula in any case of Orlicz function M.