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Existence of a solution and a positive solution of a boundary value problem for a nonlinear fourth-order dynamic equation

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Abstract

In this paper, by using the Schauder fixed point theorem, we offer some existence criteria for a solution and a positive solution to the following fourth-order two-point boundary value problem on time scale $\mathbb{T}$:

$$\begin{cases} u^{\triangle\triangle\triangle\triangle}(t) - f(t, u(t), u^{\triangle\triangle}(t)) = 0, t \in \left[a, \rho^2(b)\right], \\ u(a) = A, u(\sigma^2(b)) = B, u^{\triangle\triangle}(a) = C,\ u^{\triangle\triangle}(b) = D, \end{cases}$$

where $a, b \in \mathbb{T}$ and $a < b$.

MSC: 34B15; 39A10

Keywords: Time scale; Dynamic equation; Boundary value problem; Solution and positive solution; Fixed point

1. Introduction

The theory of dynamic equations on time scales has become a new important mathematical branch (see, for example, [1,7,14,15]) since it was initiated by Hilger [14]. At the same time, boundary value problems (BVPs) for dynamic equations on time scales have received considerable attention [2–5,8,10–13,18]. However, to the best of our knowledge, few papers can be found in the literature on BVPs for fourth-order dynamic equations on time scales.

In this paper we are interested in the existence of a solution and a positive solution for the following fourth-order BVP on time scale $\mathbb{T}$:

$$\begin{cases} u^{\triangle\triangle\triangle\triangle}(t) - f(t, u(t), u^{\triangle\triangle}(t)) = 0, t \in \left[a, \rho^2(b)\right], \\ u(a) = A, u(\sigma^2(b)) = B, u^{\triangle\triangle}(a) = C,\ u^{\triangle\triangle}(b) = D, \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{T}$ and $a < b$.

Because of its significant applications and interest to physics, the problem (1.1) has been studied previously for when $\mathbb{T} = \mathbb{R}, a = 0$ and $b = 1$ by many authors, for example, Usmani [19], Aftabizadeh [6], Yang [20], Pino

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and Manasevich [9], Ma [17], Liu [16] and the references therein. We note that all the conditions imposed on the nonlinearity f in the above-mentioned papers are overall, that is, the conditions imposed on f are on the whole domain.

Motivated by the above-mentioned papers and the main ideas in [18,21], our purpose in this paper is to consider the "height" of the nonlinear term f on a bounded set (so our conditions are local) and establish some existence criteria for a solution and a positive solution for BVP (1.1) by using the Schauder fixed point theorem. Our results are new even for the special case of difference equations.

In the remainder of this section we will provide without proof several foundational definitions and results from the calculus on time scales so that the paper is self-contained. For more details, one can see [3,5,10,11,14,15].

Definition 1.1. Let $\mathbb{T}$ be a time scale. Define the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}, \qquad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$$

for any $t \in \mathbb{T}$ (here we define $\inf \phi = \sup\ \mathbb{T}$ and $\sup \phi = \inf\ \mathbb{T}$).

Definition 1.2. Assume that $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$ (if $t = \sup\ \mathbb{T}$, we assume t is not left-scattered). Then x is called differential at $t \in \mathbb{T}$ if there exists a $\theta \in \mathbb{R}$ such that for any given $\varepsilon > 0$, there is an open neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - \theta\,[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|, \quad s \in U.$$

In this case, θ is called the delta derivative of x at $t \in \mathbb{T}$ and denoted by $\theta = x^{\triangle}(t)$.

Definition 1.3. If $X^{\triangle}(t) = x(t)$, then we define the integral by

$$\int_a^t x(s)\triangle s = X(t) - X(a).$$

Lemma 1.4. *If $x \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} x(s)\triangle s = \mu(t)x(t),$$

where $\mu(t) = \sigma(t) - t$ is the graininess function.

Lemma 1.5. *Assume that $G_1(t,s)$ $(G_2(t,s))$ is the Green's function of*

$$\begin{cases} u^{\triangle\triangle}(t) = 0, t \in [a,b], \\ u(a) = 0, u(\sigma^2(b)) = 0, \end{cases} \quad \left(\begin{cases} u^{\triangle\triangle}(t) = 0, t \in \left[a, \rho^2(b)\right], \\ u(a) = 0, u(b) = 0. \end{cases} \right)$$

Then

$$G_1(t,s) = \begin{cases} -\dfrac{(t-a)(\sigma^2(b) - \sigma(s))}{\sigma^2(b) - a}, t \le s, \\ -\dfrac{(\sigma(s) - a)(\sigma^2(b) - t)}{\sigma^2(b) - a}, t \ge \sigma(s). \end{cases} \quad \left(G_2(t,s) = \begin{cases} -\dfrac{(t-a)(b - \sigma(s))}{b-a}, t \le s, \\ -\dfrac{(\sigma(s) - a)(b - t)}{b-a}, t \ge \sigma(s). \end{cases} \right)$$

2. Main results

Throughout this paper, we use the notation

$$\mathbb{R}^+ = [0, +\infty), \qquad \mathbb{R}^- = (-\infty, 0], \qquad \eta = \max\{|A|, |B|, |C|, |D|\},$$

$$h_1 = \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} |G_1(t,s)|\triangle s, \qquad h_2 = \max_{t \in [a,b]} \int_a^{\rho(b)} |G_2(t,s)|\triangle s,$$

$$h = \max\{h_1, h_2\}.$$

In Theorem 2.1 and Corollary 2.2, we shall consider the existence of a solution for the BVP (1.1).