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A generalized vector variational inequality problem with a set-valued semi-monotone mapping

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Abstract

In this paper, we first generalize the concept of semi-monotonicity for a scalar single-valued mapping to the case of a vector set-valued mapping. Then we consider a generalized vector variational inequality problem concerning the vector set-valued semi-monotone mapping; several existence results are obtained by using the Kakutani–Fan–Glicksberg fixed point theorem which generalize some results for vector variational inequality problems studied in recent years.

MSC: 49J40; 90C29

Keywords: Set-valued mapping; Semi-monotone mapping; Generalized vector variational inequality; Kakutani–Fan–Glicksberg fixed point theorem

1. Introduction

In 1980, Giannessi [5] introduced the theory of the vector variational inequality in finite dimensional space. Since then, a lot of applications have been found. Now it is a powerful tool in the study of vector optimization and traffic equilibrium problems; see [6,9]. Due to its wide application, the theory of the vector variational inequality is generalized in different directions, and many existence results and algorithms for vector variational inequality problems have been established under various conditions; see for example [1,10] and the references therein.

It is well known that the monotonicity of a nonlinear mapping is one of the most frequently used hypotheses in the theory of the variational inequality. There are many kinds of generalizations of the monotonicity in the literature of recent years, such as pseudo-monotonicity, quasi-monotonicity, etc. In [3], Chen introduced the concept of semi-monotonicity for a single-valued mapping, which occurred in the study of nonlinear partial differential equations of divergence type. In [4], a generalization of semi-monotonicity, the so called relaxed η-α-semi-monotonicity was introduced by Y.P. Fang and a variational-like inequality problem related to it was studied.

In this paper, we generalize the semi-monotonicity for a single-valued mapping to the case of a vector set-valued mapping, and then investigate a generalized vector variational inequality problem related to this kind of vector set-valued mapping. By using the Kakutani–Fan–Glicksberg fixed point theorem, several existence results for the generalized vector variational inequality are established, which generalize the main results of [3], and enrich the theory of the vector variational inequality.

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2. Preliminaries

In this section we recall several definitions and lemmas, which will be used in the sequel.

Suppose E, F are real Banach spaces, and $L(E, F)$ is the family of all linear bounded operators from E to F, K is a nonempty convex closed subset of E. Recall that a subset C of F is said to be a closed convex cone if C is closed and $C + C \subset C$, $\lambda C \subset C$ for $\lambda > 0$. In addition, if $C \neq F$, then C is called a proper closed convex cone. A closed convex cone is pointed if $C \cap (-C) = \{0\}$. A mapping $C : K \to 2^F$ is said to be a cone mapping if $C(x)$ is a proper closed convex pointed cone and $\operatorname{int} C(x) \neq \emptyset$ for each $x \in K$.

Definition 2.1. A vector set-valued mapping $T : K \to 2^{L(E,F)}$ is said to be monotone on K if for any $x, y \in K$, it holds that

$$\langle \xi - \eta,\ y - x\rangle \in C_-, \quad \forall\, \xi \in T(x),\ \eta \in T(y),$$

where $C_- = \bigcap_{x \in K} C(x)$.

The concept of monotonicity for a vector single-valued mapping was first introduced in [2] by Chen, and then generalized by K.L. Lin et al. to the case of vector set-valued mappings; for more details see, for instance, [8].

Now, we define the concept of vector set-valued semi-monotone mapping.

Definition 2.2. A vector set-valued mapping $A : K \times K \to 2^{L(E,F)}$ is said to be a vector set-valued semi-monotone mapping on K if it satisfies the following conditions:

(1) for each $u \in K$, the mapping $A(u, \cdot) :\ K \to 2^{L(E,F)}$ is a vector set-valued monotone mapping in the sense of Definition 2.1;

(2) for each $v \in K$, the mapping $A(\cdot, v) :\ K \to 2^{L(E,F)}$ is lower semi-continuous on K, where K is equipped with the weak topology, and $L(E, F)$ is equipped with the uniform convergence topology of operators.

Now we give an example of a set-valued semi-monotone mapping.

Example. Let H be a Hilbert space, K be a nonempty subset of H, $f : H \to \mathbb{R}$ be a convex function. So $L(H, \mathbb{R}) = H$. We define a set-valued mapping as follows:

$$T : K \times K \to 2^H, \qquad T(u, v) = \partial f(v) + \|u\|\, u,$$

where $\partial f(\cdot)$ denotes the subdifferential of f in the sense of convex analysis; it is a subset of H. It is not difficult to verify that T is a set-valued semi-monotone mapping.

Remark. If $F = \mathbb{R}$ and A is single-valued, the set-valued semi-monotone mapping reduces to the semi-monotone mapping introduced by Chen in [3]. As we all know, nonlinear mappings that appear in the theory of partial differential equations are very complex. For example, some nonlinear mapping may depend on two variables, being monotone with respect to one of the variables and compact with respect to the other one. However, we cannot hope for them to be monotone or compact with respect to the two variables simultaneously. This is why we are interested in the so called semi-monotone mapping. In fact, this is not the only case: F. Browder proposed a concept of semi-accretive mapping a long time ago; for more details, see [3] and the references therein.

Vector variational inequality problems for set-valued mappings have received an increasing amount of attention in recent years. In this paper, we investigate the following generalized vector variational inequality problem (for short, the (GVVIP)).

Find $x_0 \in K$ such that for each $y \in K$, there exists $\xi \in A(x_0, x_0)$ satisfying

$$\langle \xi, y - x_0\rangle \notin -\operatorname{int} C(x_0).$$

The (GVVIP) includes many variational inequality problems as special cases. Here we give an example.

If $F = \mathbb{R}$, $C(x) = [0, +\infty)$ for all $x \in K$ and $A : K \times K \to E^*$, then the (GVVIP) reduces to the following variational inequality problem: Find $x_0 \in K$ such that

$$\langle A(x_0, x_0), y - x_0\rangle \geq 0, \quad \forall\, y \in K.$$

This is exactly the problem studied in [3].