Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 70 (2009) 1193–1208
Nonlinear Analysis
www.elsevier.com/locate/na

Existence and multiplicity for a resonance problem for the p-Laplacian on bounded domains in $\mathbb{R}^N$

Gabriel López Garza[a], Adolfo J. Rumbos[b,*]

[a] Universidad Autónoma Metropolitana, México D.F., Mexico
[b] Pomona College, Claremont, CA, USA

Received 2 June 2007; accepted 5 February 2008

Abstract

The existence of at least two solutions for a resonance problem involving the p-Laplacian is shown for the case of bounded domains in $\mathbb{R}^N$. This work constitutes an extension of a previous result of Landesman, Robinson and Rumbos for the case $p = 2$ [E. Landesman, S. Robinson, A. Rumbos, Multiple solutions of semilinear elliptic problems at resonance, Nonlinear Analysis TMA 24 (7) (1995) 1049–1059].

Keywords: Resonance; p-Laplacian; Landesman–Lazer condition; Morse index; Critical groups

1. Introduction

In this paper we study the existence of solutions of the boundary value problem

$$\begin{cases} \Delta_p u + \lambda_1 |u|^{p-2}u + g(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$;

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

is the N-dimensional p-Laplace operator with $p > 1$; λ_1 is the principal eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{2}$$

and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. the map $\xi \to g(x,\xi)$ is continuous for a.e. $x \in \Omega$, while $x \to g(x,\xi)$ is measurable on Ω for all $\xi \in \mathbb{R}$).

* Corresponding address: Pomona College, Mathematics, 610 N. College Avenue, 91711 Claremont, CA, USA. Tel.: +1 909 621 8409; fax: +1 909 607 1247.
E-mail addresses: gabl@xanum.uam.mx (G. López Garza), arumbos@pomona.edu (A.J. Rumbos).

0362-546X/$ - see front matter
doi:10.1016/j.na.2008.02.003

The eigenvalue λ_1 can be obtained as the solution to the minimization problem

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla u|^p : u \in W_o^{1,p}(\Omega) \text{ with } \int_\Omega |u|^p = 1\right\}. \quad (3)$$

Here and throughout the paper, we consider the Sobolev space $W_o^{1,p}(\Omega)$ to be the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$||u|| = \left(\int_\Omega |\nabla u|^p\right)^{1/p}.$$

The fact that λ_1 is simple and isolated goes back to works of Anane [2] in 1987 and Lindqvist [13] in 1990. Furthermore, to λ_1 there corresponds an eigenfunction, φ_1, which is in $C^{1,\nu}(\Omega)$, for some $\nu \in (0, 1)$ and which can be chosen to be positive in Ω. Moreover, according to Anane and Tsouli in [4], there exists a second eigenvalue, $\lambda_2 > \lambda_1$, of (2) obtained from

$$\lambda_2 = \inf\{\lambda \mid \lambda \text{ is an eigenvalue of (2) with } \lambda > \lambda_1\}. \quad (4)$$

It can also be shown (see Lemma 2.1) that there exists $\overline{\lambda} \in (\lambda_1, \lambda_2]$ such that

$$\overline{\lambda} \int_\Omega |w|^p \le \int_\Omega |\nabla w|^p \quad (5)$$

for all $w \in W_o^{1,p}(\Omega)$ with $\int_\Omega \varphi_1^{p-1} w = 0$.

We assume that the function $g(x, \xi)$ in (1) satisfies the following growth conditions:

$$g(x, \xi) \quad \text{is bounded for positive values of } \xi; \quad (6)$$

$$\lim_{\xi\to-\infty} \frac{g(x, \xi)}{|\xi|^{p-2}\xi} = b_o \quad \text{uniformly for a.e. } x \in \Omega, \quad (7)$$

b_o being a constant satisfying

$$0 < b_o < \overline{\lambda} - \lambda_1. \quad (8)$$

One of the existence results in this paper (see Theorem 3.2) states that if g satisfies the growth conditions (6)–(8) stated above, as well as a Landesman–Lazer type condition similar to that used in [12], then problem (1) has at least one solution. We then proceed to show in Theorem 4.1 that, in the case $p \geqslant 2$ and $g(x, \xi) = g(\xi)$ for all $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$, where

$$g(0) = 0,$$

and g is a C^1 function satisfying

$$g'(0) < 0,$$

(1) has a nontrivial solution.

The methods used in this paper are variational. In particular, we use the saddle point theorem of Rabinowitz in [15]. This yields existence of at least one solution. To get the existence of nontrivial solution in the case $p = 2$, once the saddle point is found, a theorem of Ambrosetti [1] guarantees that this is a nontrivial solution via a Morse index calculation. However, this classical Morse theory argument does not work for the case $p > 2$. Nevertheless, recent extension of Morse theory for functionals defined on Sobolev Banach spaces (which are not Hilbert spaces) due to Cingolani and Vannella in [7] and [8] can be used to distinguish the saddle point solution from the trivial one through the use of critical groups.

Related existence results to the one in this paper for bounded g may be found in the articles by Drábek and Robinson [10] and by Arcoya and Orsina [3]. In the case of multiplicity results for resonant problems with unbounded g, we note the work of Liu and Su [14], which applies to a different class of functions g from the one treated here. In the case of nonresonance, we mention the work of Cingolani and Degiovanni [6]. As noted in the abstract, the results here extend multiplicity result on bounded domains for the case $p = 2$ in [12].