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Some results on second-order neutral functional differential equations with infinite distributed delay[☆]

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Abstract

In this paper, we consider two types of second-order neutral functional differential equations with infinite distributed delay. By choosing available operators and applying Krasnoselskii's fixed-point theorem, we obtain sufficient conditions for the existence of periodic solutions to such equations.

MSC: 34K40; 34K13

Keywords: Neutral functional differential equation; Positive periodic solutions; Fixed point; Distributed delay; Second order; Operator

1. Introduction

Consider the following two types of second-order neutral functional differential equations with infinite distributed delay

$$\left(x(t) - c\int_{-\infty}^{0} K(r)x(t+r)\mathrm{d}r\right)'' = a(t)x(t) - \lambda b(t)\int_{-\infty}^{0} K(r)f(x(t+r))\mathrm{d}r, \tag{1.1}$$

and

$$\left(x(t) - c\int_{-\infty}^{0} K(r)x(t+r)\mathrm{d}r\right)'' = -a(t)x(t) + \lambda b(t)\int_{-\infty}^{0} K(r)f(x(t+r))\mathrm{d}r, \tag{1.2}$$

here λ is a positive parameter; c is a constant with $|c| < 1$; $a(t) \in C(\mathbb{R}, (0, \infty))$, $b(t) \in C(\mathbb{R}, (0, \infty))$, $a(t)$ and $b(t)$ are ω-periodic functions; $f(x) \in C(\mathbb{R}, (0, \infty))$; $K(r) \in C((-\infty, 0], [0, \infty))$ and $\int_{-\infty}^{0} K(r)\mathrm{d}r = 1$.

Neutral functional differential equations are not only an extension of ordinary delay differential equations but also provide good models in many fields including Biology, Mechanics and Economics [1,2,4,15]. For example, in

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population dynamics, since a growing population consumes more (or less) food than a matured one, depending on individual species, this leads to neutral functional equations [4]. The study on neutral functional differential equations is more intricate than ordinary delay differential equations, that is why comparing plenty of results on the existence of positive periodic solutions for various types of first-order or second-order ordinary delay differential equations, studies on positive periodic solutions for neutral differential equations are relatively less, and most of them are confined to first-order neutral differential equations, see [3,5–8,13]. Very recently, in [11], Wu and Wang discussed the second-order neutral delay differential equation

$$(x(t) - cx(t - \delta))'' + a(t)x(t) = \lambda b(t) f(x(t - \tau(t))), \tag{1.3}$$

where λ is a positive parameter, δ and c are constants with $|c| \neq 1$, $a(t), b(t) \in C(\mathbb{R}, (0, \infty))$, $f \in C([0, \infty), [0, \infty))$, and $a(t)$, $b(t)$, $\tau(t)$ are ω-periodic functions. The key step in [11] is the application of a theorem of Zhang in [14] for the neutral operator $(Ax)(t) = x(t) - cx(t - \delta)$, and the fixed-point index theorem, to obtain the existence of positive periodic solutions for (1.3) with $c < 0$.

In the paper we continue on the research of second-order neutral delay differential equation. To be concrete, we consider the equations with infinite distributed delay, i.e. (1.1) and (1.2). The delay arises from the models by employing a distributed time lag approach in which the contributions of time delay are expressed as a weighted response over a finite interval of past time through appropriately chosen memory kernels [9,10]. For (1.1) and (1.2), the said theorem of Zhang for neutral operator in [14] does not apply. To get around this, in this paper, we obtain various sufficient conditions for the existence of positive periodic solutions of (1.1) and (1.2) by employing two available operators and applying Krasnoselskii's fixed-point theorem. The techniques used are quite different from that in [11] and our results are more general than those in [11] in two aspects. First, when $c < 0$, our result enlarges the range of c in [11]. Second, we also establish results for the existence of positive solutions for (1.1) and (1.2) when $c > 0$, the case for which [11] has not been discussed. Besides, as far as we know, up to this point there has been no result on (1.1) even for the simple case $c = 0$, and in this paper we present the Green function and integrated form of (1.1) for the first time. This should be helpful for further studies in this type of equations.

2. Some lemmas

Let $X = \{x(t) \in C(\mathbb{R}, \mathbb{R}) : x(t + \omega) = x(t), t \in \mathbb{R}\}$ with norm $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$. Clearly, $(X, \|\cdot\|)$ is a Banach space. Define

$$C_\omega^+ = \{x(t) \in C(\mathbb{R}, (0, +\infty)) : x(t + \omega) = x(t)\}, \qquad C_\omega^- = \{x(t) \in C(\mathbb{R}, (-\infty, 0)) : x(t + \omega) = x(t)\}.$$

Denote

$$M = \max\{a(t) : t \in [0, \omega]\}, \qquad m = \min\{a(t) : t \in [0, \omega]\}, \qquad \beta = \sqrt{M},$$

$$A_1 = \frac{\exp(-\frac{\beta w}{2})}{\beta(1 - \exp(-\beta w))}, \qquad B_1 = \frac{1 + \exp(-\beta w)}{2\beta(1 - \exp(-\beta w))},$$

$$A_2 = \frac{\cos\frac{\beta\omega}{2}}{2\beta \sin\frac{\beta\omega}{2}}, \qquad B_2 = \frac{1}{2\beta \sin\frac{\beta\omega}{2}},$$

and

$$F(x) = b(t) f(x(t + r)) - ca(t)x(t + r).$$

The following is Krasnoselskii's fixed-point theorem which our results will be based on.

Theorem A ([12]). *Let X be a Banach space. Assume K is a bounded closed convex subset of X. If Q, S: $K \to X$ satisfy*

(i) $Qx + Sy \in K, \forall x, y \in K$,
(ii) *S is a contractive operator, and*
(iii) *Q is a completely continuous operator in K, then $Q + S$ has a fixed point in K.*