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Uniqueness of the bounded solution to a strongly degenerate parabolic problem☆

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Abstract

This paper concerns the uniqueness of the bounded solution to a strongly degenerate parabolic problem. The equation considered may have two kinds of strong degeneracies and there is no restriction on the relation between the two degeneracies. By using Holmgren's approach, we prove that the bounded solution of the associated initial–boundary value problem is unique under some essentially necessary condition on the growth of the convection.

MSC: 35K65; 35M10

Keywords: Uniqueness; Bounded solution; Strong degeneracy

1. Introduction

The purpose of the present paper is to establish a uniqueness theorem for the bounded solution to a strongly degenerate parabolic problem of the form

$$\frac{\partial \sigma(u)}{\partial t} = \frac{\partial^2 A(u)}{\partial x^2} + \frac{\partial B(u)}{\partial x}, \quad (x,t) \in Q_T = (0,1) \times (0,T), \tag{1.1}$$

$$A(u(0,t)) = g_1(t), \qquad A(u(1,t)) = g_2(t), \quad t \in (0,T), \tag{1.2}$$

$$\sigma(u(x,0)) = \sigma_0(x), \quad x \in (0,1), \tag{1.3}$$

where $\sigma(s), A(s), B(s) \in C^1(\mathbb{R})$, $\sigma'(s) \geq 0$ and $A(s)$ is strictly increasing. The strong degeneracies lie in both the set $E = \{s \in \mathbb{R} : A'(s) = 0\}$ and the set $F = \{s \in \mathbb{R} : \sigma'(s) = 0\}$ being allowed to have an infinite number of points, and in particular, the set F being allowed to have interior points.

Eq. (1.1) includes several interesting and important nonlinear partial differential equations coming from studies on fluid flowing through a porous medium, on Stefan-type problems, and so on. Numerous results on such equations have

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been established; see, e.g., the survey paper [26], the books [12,27], the bibliography [21] and the references therein, and more recently [2,3,6–8]. The most typical equations are

$$\frac{\partial\sigma(u)}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial B(u)}{\partial x} \tag{1.4}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial^2 A(u)}{\partial x^2} + \frac{\partial B(u)}{\partial x}, \tag{1.5}$$

which are of typical elliptic–parabolic type and hyperbolic–parabolic type, respectively. For the initial–boundary value problem of Eq. (1.4), the existence and uniqueness of weak solutions with $u \in L^2(0, T; H^1(0, 1))$ have been established in [1]. There, one can also find a comparison principle, however only for the so-called strong solutions, i.e. weak solutions with $\sigma(u)_t \in L^2(Q_T)$. Some other results for Eq. (1.4), including ones on the continuity of the interface between the saturated region and unsaturated region, have also been obtained; see [9,11,13,16,20,22–24, 29] for details. Research on weak solutions of Eq. (1.5) without convection dates back to the 1950s. It was Oleĭnik, et al. [19] who first introduced weak solutions with $A(u)_x \in L^\infty(Q_T)$ of the Cauchy problem of this equation and established the existence and uniqueness theorems. Studies on Eq. (1.5) are available in the survey papers [26,27] and the references therein. Among those, we refer the reader to [10,14,15,19,25,28] for the uniqueness results. In particular, Zhao [28] proved the uniqueness theorem of the bounded solution by Holmgren's approach.

Eq. (1.1) is complicated in that it may have two kinds of strong degeneracies, and it is, in fact, of mixed elliptic–hyperbolic–parabolic type. Using Kružkov's device of doubling the variables for the first-order quasilinear equation [17], Carrillo proved in [7] the uniqueness of the entropy solution for the initial–boundary value problem of Eq. (1.1) under the assumption that $B(s)$ is monotone. The author also established the existence result under some strict and complicated assumptions on $B(s)$, $\sigma(s)$ and $A(s)$. For the special case of $B(s) \equiv 0$, i.e. Eq. (1.1) without convection, we refer the reader to [3–5,2,8] for the existence and uniqueness results for weak solutions and renormalized solutions under some restrictive conditions.

In this paper, we will prove the uniqueness of the solution of the initial–boundary value problem of Eq. (1.1) in an almost largest class of solutions, namely the class consisting of bounded functions. For the typical equation (1.5), it was Zhao [28] who proved the uniqueness theorem for the bounded solution as mentioned above. However, to our knowledge, such a result even for the typical Eq. (1.4) had not been established before. Moreover, in our study, there is no restriction on the relation between $\sigma(s)$ and $A(s)$, and between $A(s)$ and $B(s)$, and no structural condition on $A(s)$. In fact, we only need the following condition on the growth of the convection:

$$|B'(s)|^2 \le h(s)\sigma'(s), \quad s \in \mathbb{R} \tag{1.6}$$

with $h(s)$ a given continuous function, which is essentially necessary for the desired uniqueness result due to the degeneracy caused by $\sigma(s)$.

The method which we use in this paper is still Holmgren's approach [27,28]. The essential idea of this approach is to reduce the proof of the uniqueness theorem to the proof of the existence of solutions to the adjoint problem. Eq. (1.1) may have strong degeneracies; so does the adjoint equation accordingly. On the other hand, since our uniqueness is just built for the bounded solution, the coefficients of the adjoint equation might not be smooth enough. For these reasons, we consider an approximate adjoint problem and establish suitable estimates on the L^2 norm and the norm of the bounded variation of its solutions. In particular, because of the existence of two kinds of degeneracies, the approximate adjoint problem is more complicated than the one with respect to (1.4) or (1.5), and we have to overcome some technical difficulties to obtain the necessary estimates and to reach the desired result. We point out that, in the view of the mathematics, the condition (1.6) is crucial for obtaining the necessary estimates for the approximate adjoint problem.

In the present paper, we only consider the problem in one dimension. For the multi-dimensional case, if we could establish the estimates on the norm of bounded variation of solutions of the adjoint problem, we might obtain the uniqueness result by the same method as in this paper. But it seems there has been no way to obtain these estimates up to now. We point out that we may establish the existence of weak solutions for such a problem utilizing the regularization method and the compensated compactness method. This result will be left to our subsequent paper [18] owing to the limitation on contents; there we establish the existence theorem even for the multi-dimensional case.