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Existence and uniqueness results for the 2-D dissipative quasi-geostrophic equation

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Abstract

This paper concerns itself with Besov space solutions of the 2-D quasi-geostrophic (QG) equation with dissipation induced by a fractional Laplacian $(-\Delta)^\alpha$. The goal is threefold: first, to extend a previous result on solutions in the inhomogeneous Besov space $B^r_{2,q}$ [J. Wu, Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces, SIAM J. Math. Anal. 36 (2004–2005) 1014–1030] to cover the case when $r = 2 - 2\alpha$; second, to establish the global existence of solutions in the homogeneous Besov space $\mathring{B}^r_{p,q}$ with general indices p and q; and third, to determine the uniqueness of solutions in any one of the four spaces: $B^s_{2,q}$, $\mathring{B}^r_{p,q}$, $L^q((0,T); B^{s+\frac{2\alpha}{q}}_{2,q})$ and $L^q((0,T); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q})$, where $s \geq 2 - 2\alpha$ and $r = 1 - 2\alpha + \frac{2}{p}$.

MSC: 35Q35; 76B03

Keywords: 2-D quasi-geostrophic equation; Besov space; Existence and uniqueness

1. Introduction

The 2-D dissipative quasi-geostrophic (QG) equation concerned here assumes the form

$$\partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \tag{1.1}$$

where $\kappa > 0$ and $\alpha \geq 0$ are parameters, $\theta = \theta(x,t)$ is a scalar function of $x \in \mathbb{R}^2$ and $t \geq 0$, and u is a 2-D velocity field determined by θ through the relations

$$u = (u_1, u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi) \quad \text{and} \quad (-\Delta)^{\frac{1}{2}}\psi = \theta. \tag{1.2}$$

The fractional Laplacian operator $(-\Delta)^\beta$ for a real number β is defined through the Fourier transform, namely

$$\widehat{(-\Delta)^\beta f}(\xi) = (2\pi|\xi|)^{2\beta}\widehat{f}(\xi)$$

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where the Fourier transform $\widehat{f}$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi}\, \mathrm{d}x.$$

For notational convenience, we write Λ for $(-\Delta)^{\frac{1}{2}}$ and combine the relations in (1.2) into

$$u = \nabla^{\perp} \Lambda^{-1}\theta,$$

where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$. Physically, (1.1) models the temperature evolution on the 2-D boundary of a 3-D quasi-geostrophic flow and is sometimes referred to as the surface QG equation [8,13].

Fundamental mathematical issues concerning the 2-D dissipative QG equation (1.1) include the global existence of classical solutions and the uniqueness of solutions in weaker senses. In the subcritical case $\alpha > \frac{1}{2}$, these issues have been more or less resolved [9,14]. When $\alpha \le \frac{1}{2}$, the issue on the global existence of classical solutions becomes extremely difficult. For the critical case $\alpha = \frac{1}{2}$, this issue was first dealt with by Constantin et al. [7] and later studied in [3,6,10,11,17] and other works. A recent work of Kiselev et al. [12] appears to have resolved this issue (in the periodic case) by removing the L^{∞}-smallness condition of [7]. Another recent progress on the critical dissipative QG equation was given in the work by Caffarelli and Vasseur [1]. We also mention other interesting investigations on related issues (see cf. [2,15,16]). The supercritical case $\alpha < \frac{1}{2}$ remains a big challenge. This paper is mainly devoted to understanding the behavior of solutions of (1.1) with $\alpha < \frac{1}{2}$. Although our attention is mainly focused on the case when $\alpha < \frac{1}{2}$, the results presented here also hold for $\alpha \ge \frac{1}{2}$. We attempt to accomplish three major goals that we now describe.

In [17], we established the global existence of solutions of (1.1) in the inhomogeneous Besov space $B^r_{2,q}$ with $1 \le q \le \infty$ and $r > 2 - 2\alpha$ when the corresponding initial data θ_0 satisfies

$$\|\theta_0\|_{B^r_{2,q}} \le C\kappa$$

for some suitable constant C. Our first goal is to extend this result to cover the case when $r = 2-2\alpha$. For this purpose, we derive a new a priori bound on solutions of (1.1) in $B^{2-2\alpha}_{2,q}$. When combined with a procedure detailed in [17], this new bound yields the global existence of solutions in $B^{2-2\alpha}_{2,q}$. As a special consequence of this result, the 2-D critical QG equation ((1.1) with $\alpha = \frac{1}{2}$) possesses a global H^1-solution if the initial datum is comparable to κ.

Our second goal is to explore solutions of (1.1) in the homogeneous Besov space $\mathring{B}^r_{p,q}$ with general indices $2 \le p < \infty$ and $1 \le q \le \infty$. This study was partially motivated by the lower bound

$$\int_{\mathbb{R}^d} |f|^{p-2} f \cdot (-\Delta)^{\alpha} f\, \mathrm{d}x \ge C 2^{2\alpha j} \|f\|^p_{L^p} \tag{1.3}$$

valid for any function f that decays sufficiently fast at infinity and satisfies

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\},$$

where $0 < K_1 \le K_2$ are constants and j is an integer. This inequality, recently established in [5,18], provides a lower bound for the integral generated by the dissipative term when we estimate solutions of (1.1) in $\mathring{B}^r_{p,q}$. Combining this lower bound with suitable upper bounds for the nonlinear term, we are able to derive a priori estimates for solutions of (1.1) in $\mathring{B}^r_{p,q}$. Applying the method of successive approximation, we then establish the existence and uniqueness of solutions emanating from initial data θ_0 satisfying

$$\|\theta_0\|_{\mathring{B}^r_{p,q}} \le C\kappa,$$

where $2 \le p < \infty$, $1 \le q \le \infty$ and $r = 1 - 2\alpha + \frac{2}{p}$. Setting $q = p$, we obtain as a special consequence the global solutions in the homogeneous Sobolev space $\mathring{W}^{p,r}$, where $2 \le p < \infty$ and $r = 1 - 2\alpha + \frac{2}{p}$.

The third goal is to determine the uniqueness of solutions of (1.1) in the spaces

$$B^s_{2,q}, \qquad \mathring{B}^r_{p,q}, \qquad L^q\left((0,T); B^{s+\frac{2\alpha}{q}}_{2,q}\right) \quad \text{and} \quad L^q\left((0,T); \mathring{B}^{r+\frac{2\alpha}{q}}_{p,q}\right) \tag{1.4}$$