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Existence of positive solutions for singular boundary value problems involving the one-dimensional p-Laplacian

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ABSTRACT

This paper studies the existence of positive solutions for singular Dirichlet boundary value problems. These results are obtained by using the Global continuation theorem, fixed point index theory and approximate method.

1. Introduction

Consider the following boundary value problem

$$\begin{cases} (\varphi_p(u'(t)))' + \lambda h(t) f(u(t)) = 0, & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{P$_\lambda$}$$

where $\varphi_p(x) = |x|^{p-2}x$, $p > 1$, $h \in C((0,1),(0,\infty))$, $f \in C([0,\infty),[0,\infty))$ with $f(u) > 0$ for $u > 0$ and λ is a nonnegative real parameter. By a positive solution of (P$_\lambda$), we mean a function $u \in C_0[0,1] \cap C^1(0,1)$ with $\varphi_p(u') \in C^1(0,1)$ satisfies (P$_\lambda$) and $u > 0$ in $(0,1)$. Many papers have been devoted to the study of the above problem with various conditions on h and f over years (see [1,2,4–13,15,16,18–38,40] and references therein).

For convenience, we give a list of hypotheses which we consider in this paper.

(H) $\int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s h(\tau)\mathrm{d}\tau\right)\mathrm{d}s < \infty$,

(F$_1$) $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = \infty$,

(F$_2$) $f_0 = \lim_{u\to 0^+} \frac{f(u)}{u^{p-1}} = 0$,

and denote $(\varphi_p(u'(t)))' = \varphi_p(u'(t))'$, $\|u\| = \max_{t\in[0,1]} |u(t)|$. The following is the first result in this work.

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Theorem 1.1. *Assume $f(0) > 0$, (H) and (F_1). Then there exists $\lambda^* > 0$ such that (P_λ) has at least two positive solutions for $\lambda \in (0, \lambda^*)$, at least one positive solution for $\lambda = \lambda^*$ and no positive solution for $\lambda > \lambda^*$.*

This problem was initiated by a singular boundary value problem arising in the near-ignition analysis of a flame structure of the form

$$\begin{cases} u''(t) + \lambda p(t)e^{u(t)} = 0, & t \in (0, 1), \\ u(0) = 0 = u(1). \end{cases}$$

It was shown in [5] via shooting methods that there exists $\lambda_0 > 0$ such that there is a positive solution for $\lambda \in (0, \lambda_0)$, there is no positive solution for $\lambda > \lambda_0$. Since then, this problem has been improved in many works (see [1,5,6,15,20,26,27,36,37] and references therein). The results can be summarized as follows. Under the hypotheses of Theorem 1.1 and the monotonicity of f (i.e., f nondecreasing), the result of Theorem 1.1 is satisfied. Without the monotonicity of f, one can obtain the result that there exist $\lambda_2 \geq \lambda_1 > 0$ such that (P_λ) has at least two positive solutions for $\lambda \in (0, \lambda_1)$, one positive solution for $\lambda \in [\lambda_1, \lambda_2]$ and no positive solution for $\lambda > \lambda_2$. This result is partial since it seems to be natural that $\lambda_1 = \lambda_2$ (see [20]). Recently, in [20], the authors proved Theorem 1.1 when $p = 2$. In this work, we extend this result to $p > 1$. We use the fixed point index theory to get the desired result. By simple calculation, we can find out that the solution space of (P_λ) is not $C_0^1[0, 1]$ because of $f(0) > 0$. This causes difficulties in calculating the fixed point index. More precisely, since the solution space for our problem is not $C_0^1[0, 1]$, we should take $C_0[0, 1]$ as the solution space. To obtain the bounded open set Ω which is used to calculate the fixed point index, we need an upper solution α such that $\alpha(0), \alpha(1) > 0$ since 0 is a lower solution for our problem. We only use the uniform continuity of f on a compact set in $[0, \infty)$ to obtain such an upper solution (for details, see Section 3).

If h does not satisfy (H), (P_λ) may or may not have a positive solution depending on the conditions of f. Note that $h(t) = t^{-a}$ satisfies (H) if $a < p$, but t^{-p} does not satisfy (H). The following theorem says that (P_λ) may have no positive solution if h does not satisfy (H).

Theorem 1.2. *Assume $f(0) > 0$. If there exist $c > 0$, $\delta \in (0, 1)$ such that $h(t) \geq ct^{-p}$ for $t \in (0, \delta)$, then (P_λ) has no positive solution for all $\lambda \geq 0$.*

Since $f(0) > 0, f(u(t))$ cannot reduce the strong singularity of t^{-p} near $t = 0$. Nonexistence of the positive solution in Theorem 1.2 is caused by the strong singularity of $t^{-p}f(u(t))$ near $t = 0$. On the other hand, if we assume (F_2), we can get the following existence result although h does not satisfy (H).

Theorem 1.3. *Assume $h(t) = t^{-p}$, (F_1) and (F_2). Then, (P_1) has at least one positive solution.*

Note that Theorem 1.3 is proved for $\lambda = 1$ not depending on f. This means that (P_λ) has at least one positive solution for all $\lambda > 0$. It is well-known that if h satisfies (H), then the corresponding operator is well-defined and completely continuous (see [35]). In general, however, we don't know if there exists the corresponding operator of (P_λ) with $h(t) = t^{-p}$. Because of this difficulty, we use the approximate method to prove Theorem 1.3 (for details, see Section 4).

This paper is organized as follows: In Section 2, we introduce well-known theorems such as the global continuation theorem, a fixed point index theorem and Hardy's inequality, etc. In Section 3, we give the proofs of Theorems 1.1 and 1.2. Finally, we give the proof of Theorem 1.3 in Section 4.

2. Preliminaries

Theorem 2.1 (*[39], Global Continuation Theorem*). *Let X be a Banach space and $\mathcal{K}$ an order cone in X. Consider*

$$x = H(\mu, x), \tag{2.1}$$

where $\mu \in \mathbb{R}_+$ and $x \in \mathcal{K}$. If $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ is completely continuous and $H(0, x) = 0$ for all $x \in \mathcal{K}$. Then $\mathcal{C}_+(\mathcal{K})$, the component of the solution set of (2.1) containing $(0, 0)$ is unbounded.

Theorem 2.2 (*[22], Generalized Picone Identity*). *Let us define*

$$l_p[y] = (\varphi_p(y'))' + b_1(t)\varphi_p(y),$$
$$L_p[z] = (\varphi_p(z'))' + b_2(t)\varphi_p(z).$$

If y and z are any functions such that $y, z, b_1\varphi_p(y'), b_2\varphi_p(z')$ are differentiable on I and $z(t) \neq 0$ for $t \in I$, the generalized Picone identity can be written as

$$\frac{d}{dt}\left\{\frac{|y|^p\varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y')\right\} \tag{2.2}$$

$$= (b_1 - b_2)|y|^p \tag{2.3}$$