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General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space

Suhong Li [a,*], Lihua Li [a], Yongfu Su [b]

[a] Department of Mathematics and Physics, Hebei Normal University of Science and Technology, Qinhuangdao Hebei 066004, PR China

[b] Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300160, PR China

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ABSTRACT

Let H be a Hilbert space and f a fixed contractive mapping with coefficient $0 < \alpha < 1$, A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Consider two iterative methods that generate the sequences $\{x_n\}$, $\{y_n\}$ by the algorithm, respectively.

$$x_n = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n \mathrm{d}s + \alpha_n \gamma f(x_n) \tag{I}$$

$$y_{n+1} = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n \mathrm{d}s + \alpha_n \gamma f(y_n) \tag{II}$$

where $\{\alpha_n\}$ and $\{t_n\}$ are two sequences satisfying certain conditions, and $\Im = \{T(s) : s \geq 0\}$ is a one-parameter nonexpansive semigroup on H. It is proved that the sequences $\{x_n\}$, $\{y_n\}$ generated by the iterative method (I) and (II), respectively, converge strongly to a common fixed point $x^* \in F(\Im)$ which solves the variational inequality

$$\langle (A - \gamma f)x^*, x^* - z\rangle \leq 0 \quad z \in F(\Im).$$

1. Introduction

Let H be a real Hilbert space, let A be a bounded linear operator on H. Throughout this paper, we always assume that A is strongly positive; that is, there is a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \forall x \in H.$$

Recall that a mapping T of H into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in H$. Recall also that a contraction on H is a self-mapping f on H such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\| \quad x, y \in H.$$

Where $\alpha \in [0, 1)$ is a constant.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [1–6] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.1}$$

where F is the fixed point set of a nonexpansive mapping T on H and b is a given point in H.

* Corresponding author.
E-mail addresses: lisuhong103@eyou.com (S. Li), suyongfu@tjpu.edu.cn (Y. Su).

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Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b \quad n \geq 0. \tag{1.2}$$

It is proved [3] (see also [4]) that the sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique solution of the minimization problem (1.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [7] introduced the viscosity approximation method for nonexpansive mappings (see [8] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n) \quad n \geq 0, \tag{1.3}$$

where $\{\sigma_n\}$ is a sequence in (0,1). It is proved [7,8] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.3) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0 \quad x \in C.$$

Recently, Giuseppe Marino and Hong Kun Xu [9] combined the iterative method (1.2) with the viscosity approximation method (1.3) and considered the following general iteration process:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n) \quad n \geq 0 \tag{1.4}$$

and proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad x \in C$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In this paper, motivated and inspired by Giuseppe Marino and Hong Kun Xu [9], we will introduced the following iterative procedures (see (1.5) and (1.6)) for the approximation of common fixed points of a one-parameter nonexpansive semigroup $\{T(s) | 0 \leq s < \infty\}$ on a nonempty closed convex subset C in a Hilbert space:

$$x_n = (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s)x_n \mathrm{d}s + \alpha_n \gamma f(x_n) \tag{1.5}$$

$$y_{n+1} = (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s)y_n \mathrm{d}s + \alpha_n \gamma f(y_n), \tag{1.6}$$

where $\{\alpha_n\}$ and $\{t_n\}$ are sequences in $[0, 1]$ and $(0, \infty)$, respectively. And their convergence theorems can be proved under some appropriate control conditions on parameter $\{\alpha_n\}$, $\{t_n\}$. Furthermore, by using these results, we obtain two mean ergodic theorems for nonexpansive mappings in Hilbert space. The results presented in this paper extend and improve the main results in Giuseppe Marino and Hong Kun Xu [9], and the methods of the proof given in this paper are also different.

Let C be a closed convex subset of a Hilbert space H. Then a family $\Im = \{T(s) | 0 \leq s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(a) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$ and $T(0) = I$;
(b) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$.
(c) the mapping $T(\cdot)x$ is continuous, for each $x \in C$.

We denote by $F(\Im)$ the set of common fixed points of $\{T(t) : t \geq 0\}$. That is, $F(\Im) = \cap_{0 \leq s < \infty} F(T(s))$.

2. Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section.

Lemma 2.1 (*Giuseppe Marino and Hong Kun Xu [9]*). *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \bar{\gamma}$.*

Lemma 2.2 (*Shimizu and Takashi [10]*). *Let C be a nonempty bounded closed convex subset of H and let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, then for any $h \geq 0$,*

$$\lim_{t \to \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s)x \mathrm{d}s - T(h) \left(\frac{1}{t} \int_0^t T(s)x \mathrm{d}s \right) \right\| = 0.$$