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The existence of countably many positive solutions for some nonlinear three-point boundary problems on the half-line

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ABSTRACT

In this paper, we study the existence of countably many positive solutions for some nonlinear singular three-point boundary problems on the half-line

$$(\phi(u')(t))' + a(t)f(t, u(t)) = 0, \quad 0 < t < +\infty,$$
$$u(0) - B_0(u'(\eta)) = 0, \qquad u'(\infty) = 0,$$

where $\phi(s) : R \to R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$ and $\eta \in (0, +\infty)$, $a : [0, +\infty) \to [0, +\infty)$ and has countably many singularities on $[0, +\infty)$. By using the fixed-point index theorem and a new fixed-point theorem in cones, the existence of countably many solutions for singular three-point boundary value problem are obtained under conditions weaker than those used by Liu and Zhang [B.F. Liu, J.H. Zhang, The existence of positive solutions for some nonlinear boundary value problems with linear mixed boundary conditions, J. Math. Anal. Appl. 309 (2005) 505–516].

1. Introduction

The existence and multiplicity of positive solutions for linear and nonlinear ordinary differential equations and difference equations have been studied extensively. To identify a few, we refer the reader to see [2,3,5–10,15]. Recently, Kaufmann and Kosmatov [5] studied the following singular boundary-value problems under the condition that $f \in C([0, +\infty), [0, +\infty))$, $a(t) \in L^p[0, 1]$ and has countably many singularities on $(0, \frac{1}{2})$

$$\begin{cases} -u''(t) = a(t)f(u(t)), & 0 < t < 1, \\ u(0) = u(1) = 0. \end{cases}$$

They showed that there exist countably many positive solutions by using Hölder's inequality and the fixed-point theorem for an operator on a cone.

Lian [13] studied the following boundary value problem of second order differential equation with a p-Laplacian operator on a half-line

$$\begin{cases} (\varphi_p(u'(t)))' + \phi(t)f(t, u(t), u'(t)) = 0, & 0 < t < +\infty, \\ \alpha u(0) - \beta u'(0) = 0, \quad u'(\infty) = 0. \end{cases}$$

They showed the existence of at least three positive solutions by using a fixed-point theorem in a cone due to Avery–Peterson.

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In [2], Liu and Zhang studied the existence of positive solutions of quasilinear differential equation

$$(\varphi(x'))' + a(t)f(x(t)) = 0, \quad 0 < t < 1,$$
$$x(0) - \beta x'(0) = 0, \quad x(1) + \delta x'(1) = 0.$$

They obtained the existence of one or two positive solutions by using a fixed-point index theorem in cones. They assume that $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$ under conditions which are listed as follows:

(1) if $x \le y$, then $\varphi(x) \le \varphi(y)$, for all $x, y \in R$;
(2) φ is a continuous bijection and its inverse mapping is also continuous;
(3) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in [0, +\infty)$.
In the above definition, we can replace the condition (3) by the following stronger condition:
(4) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in R$, where $R = (-\infty, +\infty)$.

Remark 1.1. If conditions (1), (2) and (4) hold, then it implies that φ is homogenous generating a p-Laplace operator, i.e., $\varphi(x) = |x|^{p-2}x$, for some $p > 1$.

However, condition (3) seems too strong, in this paper we replace the condition (3) by the following weaker condition: (3$'$) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in [0, S]$, for some $S > 0$.

Motivated by all the work mentioned above, the purpose of this paper is to study the existence of countably many positive solutions of the following boundary value problems on the half-line

$$(\phi(u'))' + a(t)f(t, u(t)) = 0, \quad 0 < t < +\infty, \tag{1.1}$$

$$u(0) - B_0(u'(\eta)) = 0, \quad u'(\infty) = 0, \tag{1.2}$$

where $\phi(s) : R \to R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$ and $\eta \in (0, +\infty)$, $a : [0, +\infty) \to [0, +\infty)$ and has countably many singularities on $[0, +\infty)$. B_0 is a function and satisfies the conditions that there are nonnegative numbers B, A such that $Bx \le B_0(x) \le Ax$ and $B_0(x) - B_0(y) \le B_0(x - y)$, for $x, y \in R$.

Recently, there are few works consider the existence of countably many positive solutions. In [5–7] they obtain countably many positive solutions by assuming $a(t)$ has infinitely many singularities in $(0, \frac{1}{2})$. To the authors' knowledge, no one has studied the existence of positive solutions when $a(t)$ has infinitely many singularities in $[0, +\infty)$. The goal of present paper is to fill the gap in this area.

In the past few years there have been many papers investigated the positive solutions of boundary value problem on the half-line, see [1,11–14]. They discuss the existence and multiplicity (at least three) positive solutions to nonlinear differential equations. So far, however, there have been no papers on the existence of countably many positive solutions to boundary value problems of differential equation on infinite intervals.

In this paper, we use the fixed-point index theory and a new fixed point theorem in cones to investigate the existence of countably many solutions to boundary value problems (1.1) and (1.2).

We assume that some of the following conditions are satisfied throughout this paper:

(C_1) $f \in C([0, +\infty)^2, [0, +\infty))$, $f(t, 0) \not\equiv 0$ on any subinterval of $[0, +\infty)$ and, when u is bounded, $f(t, (1 + t)u)$ is bounded on $[0, +\infty)^2$;

(C_2) There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $1 \le t_{i+1} < t_i$, $\lim_{i\to\infty} t_i = t_0 < +\infty$ and $t_0 > 1$, $\lim_{t\to t_i} a(t) = \infty$, $i = 1, 2, \ldots,$ and

$$0 < \int_0^{+\infty} a(t)\mathrm{d}t < +\infty, \int_0^{+\infty} \phi^{-1}\left(\int_\tau^{+\infty} a(s)\mathrm{d}s\right)\mathrm{d}\tau < +\infty. \tag{1.3}$$

(C_3) There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $0 < t_{i+1} < t_i < 1$, $\lim_{i\to\infty} t_i = t_0 < +\infty$ and $0 < t_0 < 1$, $\lim_{t\to t_i} a(t) = \infty$, $i = 1, 2, \ldots,$ and (1.3) holds.

Moreover $a(t)$ does not vanish identically on any subinterval of $[0, +\infty)$.

The plan of the paper is as follows. In Section 2 we give some definitions for the convenience of the reader. In Section 3 we present some lemmas in order to prove our main results. Section 4 is devoted to presenting and proving our main results. In Section 5 we present the example of the increasing homeomorphism and positive homomorphism operators.

2. Some definitions and fixed point theorems

In this Section we provide some background definitions cited from cone theory in Banach spaces.

Definition 2.1. Let $(E, \|.\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied:

(a) if $y \in P$ and $\lambda \ge 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\le$, that is, $x \le y$ if and only if $y - x \in P$.