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Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces

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ABSTRACT

Given a uniform space X and nonempty subsets A and B of X, we introduce the concepts of some families $\mathcal{V}$ of generalized pseudodistances on X, of set-valued dynamic systems of relatively quasi-asymptotic contractions $T : A \cup B \to 2^{A\cup B}$ with respect to $\mathcal{V}$ and best proximity points for T in $A \cup B$, and we describe the methods which we use to establish the conditions guaranteeing the existence of best proximity points for T when T is cyclic (i.e. $T : A \to 2^B$ and $T : B \to 2^A$) or when T is noncyclic (i.e. $T : A \to 2^A$ and $T : B \to 2^B$). Moreover, we establish conditions guaranteeing that for each starting point each generalized sequence of iterations of these contractions (in particular, each dynamic process) converges and the limit is a best proximity point for T in $A\cup B$. These best proximity points for T are determined by unique endpoints in $A \cup B$ for a map $T^{[2]}$ when T is cyclic and for a map T when T is noncyclic. The results and the methods are new for set-valued and single-valued dynamic systems in uniform, locally convex, metric and Banach spaces. Various examples illustrating the ideas of our definitions and results, and fundamental differences between our results and the well-known ones are given.

1. Introduction

Let A and B be nonempty subsets of a metric space (X, d) and let $\mathrm{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. The single-valued map $T : A \cup B \to A \cup B$ is called *cyclic* if $T(A) \subset B$ and $T(B) \subset A$. Recall that if T is cyclic, then a point $w \in A \cup B$ is called a *best proximity point* for T if $d(w, T(w)) = \mathrm{dist}(A, B)$. The single-valued map $T : A \cup B \to A \cup B$ is called *noncyclic* if $T(A) \subset A$ and $T(B) \subset B$. If T is noncyclic, then a point $(u, v) \in A \times B$ is called a *best proximity point* for T if $T(u) = u$, $T(v) = v$ and $d(u, v) = \mathrm{dist}(A, B)$. For details, see [1,3–6].

The results concerning the existence of best proximity points were established by: Eldred, Kirk and Veeramani [3] for relatively nonexpansive cyclic and noncyclic maps $T : A \cup B \to A \cup B$ in uniformly convex Banach spaces and in Banach spaces such that the pair (A, B) has a proximal normal structure; A.A. Eldred and P. Veeramani [4] for cyclic contraction $T : A \cup B \to A \cup B$ of the Banach type in metric and uniformly convex Banach spaces X; and Di Bari, Suzuki and Vetro [1] for cyclic contractions $T : A\cup B \to A\cup B$ of the Meir–Keeler type in uniformly convex Banach spaces. Additionally, in papers [4,1],

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the uniqueness of best proximity points, the convergence to these best proximity points of every sequence $\{w^{2m}\}$ or $\{w^{2m+1}\}$ where $w^m = T^{[m]}(w^0)$ for $m \in \{0\} \cup \mathbb{N}$ and $w^0 \in A \cup B$, and relations between best proximity points of T in $A \cup B$ and fixed points of $T^{[2]}$ in $A \cup B$ were proved.

It is natural to ask whether there are some results of the above type concerning set-valued dynamic systems in uniform spaces. The main aim of this paper is to show that the answer is affirmative.

For a given uniform space X and nonempty subsets A and B of X, we introduce the concepts of some families $\mathcal{V}$ of generalized pseudodistances on X, the set-valued dynamic systems of relatively quasi-asymptotic contractions $T : A \cup B \to 2^{A\cup B}$ with respect to $\mathcal{V}$ and best proximity points for T in $A \cup B$, and we describe the methods which we use to establish the conditions guaranteeing the existence of best proximity points for T when T is cyclic (i.e. $T : A \to 2^B$ and $T : B \to 2^A$) or when T is noncyclic (i.e. $T : A \to 2^A$ and $T : B \to 2^B$). Moreover, we establish conditions guaranteeing that for each starting point each generalized sequence of iterations of these contractions (in particular, each dynamic process) converges and the limit is a best proximity point for T in $A \cup B$. These best proximity points for T are determined by unique endpoints in $A \cup B$ for a map $T^{[2]}$ when T is cyclic and for a map T when T is noncyclic. The results and the methods are new for set-valued and single-valued dynamic systems in uniform, locally convex, metric and Banach spaces. Various examples illustrating the ideas of our definitions and results, and a fundamental differences between our results and the well-known ones are given.

2. Definitions, notations and statement of results

To describe our results we need some definitions and notations.

Assume once and for all that X is a Hausdorff uniform space with uniformity defined by a saturated family $\{d_\alpha : \alpha \in \mathcal{A}\}$ of pseudometrics d_α, $\alpha \in \mathcal{A}$, uniformly continuous on X^2. Recall that a *set-valued dynamic system* is defined as a pair (X, T), where X is a certain space and T is a set-valued map $T : X \to 2^X$; in particular, a set-valued dynamic system includes the usual dynamic system where T is a single-valued map. For $T : E \to 2^X$, $E \subset X$, let $T(E) = \bigcup_{x\in E} T(x)$. Here 2^X denotes the family of all nonempty subsets of a space X.

Definition 2.1. Let X be a Hausdorff uniform space and let A and B be nonempty subsets of X. (a) $(A \cup B, T)$ is called a *cyclic set-valued dynamic system on* $A \cup B$ if $T : A \to 2^B$ and $T : B \to 2^A$. (b) $(A \cup B, T)$ is called a *noncyclic set-valued dynamic system on* $A \cup B$ if $T : A \to 2^A$ and $T : B \to 2^B$.

Definition 2.2. Let X be a Hausdorff uniform space. The family $\mathcal{V} = \{V_\alpha : 2^X \to [0, \infty], \alpha \in \mathcal{A}\}$ is said to be a *$\mathcal{V}$-semifamily of generalized pseudodistances on X* (*$\mathcal{V}$-semifamily*, for short) if the following two conditions hold:

($\mathcal{V}$1) $\forall_{\alpha\in\mathcal{A}}\forall_{E_1,E_2\in 2^X}\{E_1 \subset E_2 \Rightarrow V_\alpha(E_1) \le V_\alpha(E_2)\}$; and
($\mathcal{V}$2) $\exists_{\alpha_0\in\mathcal{A}}\{V_{\alpha_0}(X) > 0\}$.

Let $\mathcal{V} = \{V_\alpha : 2^X \to [0, \infty], \alpha \in \mathcal{A}\}$ be a $\mathcal{V}$-semifamily and, for each $\alpha \in \mathcal{A}$, let $D_{\mathcal{V};\alpha}(E_1, E_2) = \inf\{V_\alpha(\{x, y\}) : x \in E_1, y \in E_2\}$, $E_1, E_2 \in 2^X$.

A point $w \in X$ is said to be an *endpoint* (or a *stationary point*) of T if w is a fixed point of T (i.e., $w \in T(w)$) and $T(w) = \{w\}$. Now let us introduce the notion of best proximity points for cyclic and noncyclic set-valued dynamic systems in uniform spaces.

Definition 2.3. Let X be a Hausdorff uniform space, let $\mathcal{V} = \{V_\alpha : 2^X \to [0, \infty], \alpha \in \mathcal{A}\}$ be a $\mathcal{V}$-semifamily and let A and B be nonempty subsets of X.

(a) Let $(A \cup B, T)$ be a cyclic set-valued dynamic system on $A \cup B$. A point $w \in A \cup B$ is called a *best proximity point for T* if $T(w)$ is a *singleton* (i.e. $T(w) = \{T(w)\}$) and, for each $\alpha \in \mathcal{A}$, $V_\alpha(\{w, T(w)\}) = D_{\mathcal{V};\alpha}(A, B)$.
(b) Let $(A \cup B, T)$ be a noncyclic set-valued dynamic system on $A \cup B$. A point $(u, v) \in A \times B$ is called a *best proximity point for T* if u and v are endpoints of T and, for each $\alpha \in \mathcal{A}$, $V_\alpha(\{u, v\}) = D_{\mathcal{V};\alpha}(A, B)$.

It is natural to ask the following:

Question 2.1. *Let $(A \cup B, T)$ be a cyclic or noncyclic set-valued dynamic system in uniform space X. Are there any conditions guaranteeing the existence of best proximity points for T such that the convergence property holds?*

The following concept of relatively quasi-asymptotic contractions is needed to present our results which are the answer to Question 2.1.

Definition 2.4. Let X be a Hausdorff uniform space and let $\mathcal{V} = \{V_\alpha : 2^X \to [0, \infty], \alpha \in \mathcal{A}\}$ be a $\mathcal{V}$-semifamily on X. Let A and B be nonempty subsets of X and let $(A \cup B, T)$ be a set-valued dynamic system on $A \cup B$. We say that $(A \cup B, T)$ is a *$\mathcal{V}$-relatively quasi-asymptotic contraction on $A \cup B$* (*$\mathcal{V}$-RQAC on $A \cup B$*, for short) if the following two conditions hold:

(A1) $\forall_{\alpha\in\mathcal{A}}\{V_\alpha(A \cup B) > D_{\mathcal{V};\alpha}(A, B) \Rightarrow \exists_{m\in\mathbb{N}}\{V_\alpha(T^{[m]}(A \cup B)) < V_\alpha(A \cup B)\}\}$; and
(A2) $\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{\eta>0}\forall_{n\in\mathbb{N}}\{V_\alpha(T^{[n]}(A \cup B)) < D_{\mathcal{V};\alpha}(A, B) + \varepsilon + \eta \Rightarrow \exists_{m\in\mathbb{N}}\{V_\alpha(T^{[m+n]}(A \cup B)) \le D_{\mathcal{V};\alpha}(A, B) + \varepsilon\}\}$.

We now state the first main result.