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Multiple solutions of singular three-point boundary value problems on $[0, \infty)$

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ABSTRACT

In this paper, we study the existence of single and multiple solutions of three-point boundary value problems for the following nonlinear singular second-order differential equations

$$\begin{cases} x''(t) - px'(t) - qx(t) + h(t)f(t, x(t)) = 0, & t \in (0, +\infty), \\ ax(0) - bx'(0) - kx(\xi) = c \geq 0, & \lim\limits_{t \to +\infty} \dfrac{x(t)}{e^{rt}} = d \geq 0, \end{cases}$$

where $p, b \geq 0, a > k > 0, q > 0, 0 < \xi < +\infty, r \in \left[0, \frac{p+\sqrt{p^2+4q}}{2}\right]$, $h : (0, +\infty) \to (0, +\infty)$ is continuous and may be singular at $t = 0$, $f : [0, +\infty) \times [0, +\infty) \to (-\infty, +\infty)$ is continuous and may take a negative value. By applying the technique of lower and upper solutions and the theory of topological degree, we obtain the conditions for the existence of at least one solution and at least three solutions respectively.

1. Introduction

The purpose of this paper is to establish the conditions for the existence of single and multiple solutions for the following nonlinear singular second-order three-point boundary value problem (BVP, for short) on the half-line $[0, \infty)$

$$\begin{cases} x''(t) - px'(t) - qx(t) + h(t)f(t, x(t)) = 0, & t \in \mathbb{R}_0^+, \\ ax(0) - bx'(0) - kx(\xi) = c \geq 0, & \lim\limits_{t \to +\infty} \dfrac{x(t)}{e^{rt}} = d \geq 0, \end{cases} \tag{1.1}$$

where $p, b \geq 0, a > k > 0, q > 0, 0 < \xi < +\infty$ and $r \in \left[0, \frac{p+\sqrt{p^2+4q}}{2}\right]$, $h \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ may be singular at $t = 0$, $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ may take a negative value, in which $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_0^+ = (0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$.

Boundary value problems on the half-line arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena, and a lot of work has been done in this area [1,3–5,7,16–19]. In [4], Bosiud, by applying a diagonalization procedure, established the existence of bounded solutions for the following BVP on the half-line

$$\begin{cases} x''(t) + h(t)f(t, x(t)) = 0, & t \in (a, +\infty), \\ x(a) = 0, & \lim\limits_{t \to +\infty} x'(t) = 0. \end{cases} \tag{1.2}$$

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In [7], Chen and Zhang obtained some sufficient and necessary conditions for the existence of positive solutions for BVP (1.2). Recently, O'Regan, Yan and Agarwal [17] established the existence of one and two unbounded positive solutions for BVP (1.2) by employing the technique of lower and upper solutions and the theory of fixed point index. Hao, Liang and Xiao [9] studied the following second-order boundary value problem on the half-line

$$\begin{cases} x''(t) - k^2x(t) + m(t)f(t, x(t)) = 0, \quad t \in \mathbb{R}_0^+, \\ x(0) = 0, \qquad \lim_{t\to+\infty} x(t) = 0, \end{cases} \tag{1.3}$$

where $k > 0, f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+), m \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ may be singular at $t = 0$. The conditions for the existence of at least one positive solution and at least two positive solutions for BVP (1.3) are obtained by using the Krasnosel'skii–Guo theorem.

Inspired by the work of the above papers and many known results in [8,10–15], the aim of the present paper is to establish some simple criteria for the existence of single solution and three solutions of BVP (1.1). Compared to the results in [1–19], our work presented in this paper has the following new features. Firstly, BVP (1.1) possesses singularity, that is, $h(t)$ is allowed to be singular at $t = 0$. Secondly, BVP (1.1) includes the first-order derivative in addition to a singular term, which brings about many difficulties in our work. Thirdly, the nonlinear term f may take a negative value. In addition, we should mention that the results we obtained are for the existence of at least one solution and at least three solutions of BVP (1.1), while the work in [9] is for the existence of at least one and at least two solutions for BVP (1.3). The main tool used in this paper is the technique of lower and upper solutions and the theory of topological degree.

The rest of the paper is organized as follows: Some preliminaries and a number of lemmas useful to the derivation of the main results are given in Section 2, then the proofs of the main theorems are given in Section 3.

2. Preliminaries and lemmas

In this section, we present some preliminaries and lemmas that will be used in the proof of our main results. Let r_1 and r_2 be the roots of the polynomial $P(\lambda) = \lambda^2 - p\lambda - q$, namely

$$r_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \qquad r_2 = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

It is easy to see that $r_1 > 0$ and $r_2 < 0$. Let

$$C_1 = \left\{ x : \mathbb{R}^+ \to \mathbb{R} \mid x \text{ is continuous on } \mathbb{R}^+ \text{ and } \lim_{t\to+\infty} x(t) \text{ exists} \right\}.$$

From [8] we know that C_1 is a Banach space with the norm $\|x\|_1 = \sup_{t\in\mathbb{R}^+} |x(t)|$ for any $x \in C_1$. Now we define

$$C_\infty = \left\{ x : \mathbb{R}^+ \to \mathbb{R} \mid x(t) \text{ is continuous on } \mathbb{R}^+ \text{ and } \lim_{t\to+\infty} \left\{ \mathrm{e}^{-r_1 t}|x(t)| \right\} \text{ exists} \right\}.$$

For $x \in C_\infty$, define $\|x\|_\infty = \sup_{t\in\mathbb{R}^+} \left\{ \mathrm{e}^{-r_1 t}|x(t)| \right\}$. Then C_∞ is a Banach space with the norm $\|\cdot\|_\infty$, see [2].

Lemma 2.1 ([6]). *Let $M \subseteq C_1$. Then M is relatively compact in C_1 if the following conditions hold:*

(1) *M is bounded in C_1.*

(2) *the functions belonging to M are equicontinuous on any finite interval of $\mathbb{R}^+$.*

(3) *the functions from M are equiconvergent at $+\infty$, that is, for any given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.*

Lemma 2.2. *Let $M \subseteq C_\infty$. Then M is relatively compact in C_∞ if the following conditions hold:*

(1) *M is bounded in C_∞.*

(2) *the functions belonging to $\left\{ y \mid y(t) = \mathrm{e}^{-r_1 t}x(t), x \in M \right\}$ are equicontinuous on any finite interval of $\mathbb{R}^+$.*

(3) *the functions from $\left\{ y \mid y(t) = \mathrm{e}^{-r_1 t}x(t), x \in M \right\}$ are equiconvergent at $+\infty$.*

Proof. It is easy to see that $M' = \left\{ y \mid y(t) = \mathrm{e}^{-r_1 t}x(t), x \in M \right\} \subseteq C_1$ satisfies the conditions of Lemma 2.1. So there exists a sequence $\{y_n\} \subseteq M'$ and $y_0 \in C_1$ such that $\lim_{n\to\infty} \|y_n - y_0\|_1 = 0$. Let $x_n(t) = \mathrm{e}^{r_1 t}y_n(t)$ $(n = 1, 2, \cdots)$ and $x_0(t) = \mathrm{e}^{r_1 t}y_0(t)$. Obviously, $\{x_n\} \subseteq M$, $x_0 \in C_\infty$ and $\lim_{n\to\infty} \|x_n - x_0\|_\infty \leq \lim_{n\to\infty} \|y_n - y_0\|_1 = 0$. □

For convenience, we make the following assumption:

(H_0) $a - br_1 - k\mathrm{e}^{r_1\xi} > 0$.