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A contraction mapping proof of the smooth dependence on parameters of solutions to Volterra integral equations

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ABSTRACT

We consider the linear Volterra equation

$$x(t) = a(t) - \int_0^t K(t,s)\,x(s)\,\mathrm{d}s$$

and suppose that the kernel K and forcing function a depend on some parameters $\epsilon \in \mathbb{R}^d$. We prove that, under suitable conditions, the solutions depend on ϵ as smoothly the functions a and K. The proof is based on the *contraction mapping principle* and the *variational equation*. Though our conditions are not the most generally possible, they nonetheless include many important examples.

1. Introduction

We consider linear Volterra equations of the form

$$x(t) = a(t) - \int_0^t K(t,s)x(s)\,\mathrm{d}s.$$

There are three basic questions to be addressed when confronted with such an equation: does a solution exist, is that solution unique, and is the solution stable?

There are several approaches to proving the existence of solutions to differential and integral equations. Most fall into the categories of contraction mapping arguments, compactness arguments, or index theory arguments.

Of these the contraction mapping approach is probably the most elementary and has several advantages; the contraction mapping principle automatically gives uniqueness of the solution in some class and typically gives continuous dependence of the solution on the defining data.

There are many different forms of stability that have been defined for differential equations. Perhaps the weakest is the continuous dependence of the solution on the functions a and K that define the equation. This has been widely studied, see for example [1] or [2]. The work of Artstein gives a fairly complete picture of the problem of continuous dependence of solutions to Volterra integral equations [3,4]. Continuous dependence can be thought of as weak stability under misspecification of the model.

However, in many problems arising out of physics the model is certain and any uncertainty in the model specification comes from uncertainty in the measurement of various physical parameters describing the model. Often this reduces our uncertainty to some finite dimensional vector space of parameters. In this case it is the natural to consider differentiable dependence of the solution on this finite dimensional space of parameters.

We prove such differential dependence on parameters in Theorem 8. A version of our theorem can be found in [5, Theorem 1.2 Chapter 13]. It comes as a corollary of a stronger theorem proved using compactness arguments.

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2. Existence and uniqueness of solutions

We begin here by giving a version of the existence and uniqueness of solutions to the linear Volterra integral equation

$$x(t) = a(t) - \int_0^t K(t,s)x(s)\,\mathrm{d}s. \tag{1}$$

As with differential equations, an existence and uniqueness theorem can be obtained using the contraction mapping principle.

Theorem 1 (*Contraction Mapping Principle*). *Let (X, d) be a complete metric space and $P : X \to X$ be a contraction mapping, i.e. there exists $0 \le \lambda < 1$ such that*

$$d(P(x), P(y)) \le \lambda d(x, y).$$

Then P has a unique fixed point, which we denote by $\omega(P)$. Moreover, for any $x \in X$ we have

$$d(x, \omega(P)) < \frac{d(x, P(x))}{1-\lambda}.$$

First we wish to prove an existence and uniqueness theorem for finite time horizons. We denote the space of continuous functions with values in the normed space $(V, \|\cdot\|)$ on a compact metric space X by $C(X, V)$. We endow this space with the norm $\|\phi\|_0 = \sup_{x\in X}\|\phi(x)\|$ and write d_0 for the associated metric. We will use $V = \mathbb{R}^n$ endowed with the usual Euclidean norm and $V = M_n(\mathbb{R})$, the space of $n \times n$ matrices with entries in $\mathbb{R}$, endowed with the standard operator norm. For an interval $I \subseteq \mathbb{R}$ denote $\triangle(I) := \{(t, s) : s, t \in I, s \le t\}$.

Theorem 2 (*Existence and Uniqueness for a Finite Time Horizon*). *Fix $T > 0$. Suppose $a \in C([0, T], \mathbb{R}^n)$ and $K : \triangle([0, T]) \to M_n(\mathbb{R})$ such that for every $\phi \in C([0, T], \mathbb{R}^n)$*

$$\int_0^t K(t,s)\phi(s)\,\mathrm{d}s \tag{2}$$

is a continuous function of t and there exists $\lambda < 1$ such that

$$\sup_{t\in[0,T]}\int_0^t \|K(t,s)\|\,\mathrm{d}s < \lambda. \tag{3}$$

Then there exists a unique bounded solution x to (1) and $x \in C([0, T], \mathbb{R}^n)$.

Remark 1. If we have $K \in C(\triangle([0, T]), M_n(\mathbb{R}))$ then our condition (2) holds. However our kernel

$$K(t,s) = \begin{cases} \dfrac{1}{1+(t-s)} & t-1 \le s \le t \\ 0 & s < t-1 \end{cases}$$

satisfies (2) even though it is not continuous.

Proof. Define the map $P : C([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}^n)$ by

$$P(\phi)(t) := a(t) - \int_0^t K(t,s)\phi(s)\,\mathrm{d}s. \tag{4}$$

We have

$$\begin{aligned} d_0(P(\phi_1), P(\phi_2)) &\le \sup_{t\in[0,T]}\left\|\int_0^t K(t,s)(\phi_1(s)-\phi_2(s))\,\mathrm{d}s\right\| \\ &\le \sup_{t\in[0,T]}\int_0^t \|K(t,s)\|\,\|\phi_1(s)-\phi_2(s)\|\,\mathrm{d}s \\ &\le \sup_{t\in[0,T]}\int_0^t \|K(t,s)\|\,\mathrm{d}s \sup_{t\in[0,T]}\|\phi_1(t)-\phi_2(t)\| \\ &< \lambda\, d_0(\phi_1, \phi_2) \end{aligned}$$

where λ is from (3). Thus P is a contraction mapping on $C([0, T], \mathbb{R}^n)$. Thus we get a unique fixed point $x(t)$ in $C([0, T], \mathbb{R}^n)$. If we apply our estimate from the contraction mapping principle to the function 0 we get

$$\|x\|_0 = d_0(0, x) \le \frac{d_0(0, P(0))}{1-\lambda} = \frac{\|a\|_0}{1-\lambda}.$$