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Lipschitz continuity of copulas w.r.t. L_p-norms

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ABSTRACT

In the framework of the stability analysis of real functions, the Lipschitz continuity of copulas w.r.t. L_p-norms is investigated. Emphasis is put on 1-Lipschitz continuity, as it is the strongest type possible for copulas. After illustrating how to identify 1-Lipschitz copulas w.r.t. some L_p-norm, the preservation of this 1-Lipschitz continuity by certain construction methods, such as ordinal sums and convex sums, is demonstrated. Special attention is given to three important classes of copulas, namely Archimedean copulas, extreme value copulas and Archimax copulas.

1. Introduction

A standard way of expressing the stability of a continuous real function w.r.t. perturbations of its arguments is to compute its Lipschitz constant w.r.t. some L_p-norm ($p \geq 1$). For $\mathbf{x} \in \mathbb{R}^n$, its L_p-norm is defined by $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, for $p \in [1, \infty[$, and $\|\mathbf{x}\|_\infty = \max(|x_1|, \ldots, |x_n|)$. A real function $f : E \to \mathbb{R}$, defined on a convex compact subset $E \subseteq \mathbb{R}^n$, is called k-Lipschitz continuous w.r.t. some L_p-norm, $k \geq 0$, or shortly, f is k-p-Lipschitz, if for any $\mathbf{x}, \mathbf{y} \in E$ it holds that

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq k \cdot \|\mathbf{x} - \mathbf{y}\|_p .$$

Obviously, if f is k-p-Lipschitz, then it is also k-p'-Lipschitz for any $1 \leq p' \leq p$ and k'-p-Lipschitz for any $k' \geq k$. We say that k^* is the best Lipschitz constant of f w.r.t. some L_p-norm, if k^* is the smallest constant k for which f is k-p-Lipschitz, i.e. $k^* = \min\{k \in [0, \infty[\mid f \text{ is } k\text{-}p\text{-Lipschitz}\}$. This smallest Lipschitz constant is also called the L_p-norm of f and denoted by $\|f\|_p = k^*$. Note that for a function f of one argument ($n = 1$) the value of p is immaterial, and we simply write $\|f\|$.

A class of real functions enjoying increasing attention is the class of aggregation functions, i.e. increasing n-ary operations on the unit interval, satisfying some additional conditions depending on the field of study or application; see e.g. [1–4]. Important examples are triangular norms, quasi-copulas, copulas, quasi-arithmetic means, OWA operators, and so on. For applications, the stability of such aggregation functions is of utmost importance. For several subclasses of aggregation functions, such a stability analysis has already been conducted [5–11]. However, the class of copulas, the subclass of aggregation functions that are studied most nowadays, because of their important role in the modelling of the dependence structure of random vectors [1,12,13], has not been subjected to such an investigation. The purpose of our paper is to fill this gap, at least partially.

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In the next section, we recall some basic concepts and present a few initial observations. In Section 3, we explain how to identify 1-p-Lipschitz copulas and provide ample illustrations. In Section 4, we show that several well-known construction methods yield a 1-p-Lipschitz copula when departing from copulas of the same type. Each of Sections 5–7 is dedicated to an important class of copulas, more specifically Archimedean copulas, extreme value copulas and Archimax copulas.

2. Copulas

Of particular interest to our discussion are binary aggregation functions, i.e. increasing functions $A : [0, 1]^2 \to [0, 1]$ that satisfy $A(0, 0) = 0$ and $A(1, 1) = 1$ [3]. The smallest and greatest binary aggregation functions that are 1-p-Lipschitz are Yager's triangular norm $T_p^{\mathbf{Y}}$ and triangular conorm $S_p^{\mathbf{Y}}$ [14], respectively, as is shown in [15]:

$$T_p^{\mathbf{Y}}(x, y) = \begin{cases} \max\left(0, 1 - ((1-x)^p + (1-y)^p)^{1/p}\right), & \text{if } p < \infty, \\ \min(x, y), & \text{if } p = \infty, \end{cases} \tag{1}$$

and

$$S_p^{\mathbf{Y}}(x, y) = \begin{cases} \min\left(1, (x^p + y^p)^{1/p}\right), & \text{if } p < \infty, \\ \max(x, y), & \text{if } p = \infty. \end{cases} \tag{2}$$

Note that if a binary aggregation function A has a neutral element $e \in [0, 1]$, i.e. $A(x, e) = A(e, x) = x$ for any $x \in [0, 1]$, then its L_p-norm is at least 1, for any $p \in [1, \infty]$. This applies in particular to the smallest and greatest examples above, $T_p^{\mathbf{Y}}$ having neutral element 1 and $S_p^{\mathbf{Y}}$ having neutral element 0.

Definition 2.1 (*[16]*). A binary aggregation function $C : [0, 1]^2 \to [0, 1]$ is called a copula if it has neutral element 1 and is 2-increasing, i.e.

$$V_C([x, x'] \times [y, y']) = C(x', y') + C(x, y) - C(x, y') - C(x', y) \geq 0, \tag{3}$$

for any $0 \leq x \leq x' \leq 1$ and $0 \leq y \leq y' \leq 1$.

$V_C([x, x'] \times [y, y'])$ is called the C-volume of the rectangle $[x, x'] \times [y, y']$.

For any copula C it holds that $W \leq C \leq M$, with W and M the copulas defined by $W(x, y) = \max(0, x + y - 1)$ and $M(x, y) = \min(x, y)$. The Yager triangular norms are also copulas, in particular $W = T_1^{\mathbf{Y}}$ and $M = T_\infty^{\mathbf{Y}}$. The family of Yager copulas is referred to as family number 2 in [13].

A copula C is called singular if $\frac{\partial^2 C(x,y)}{\partial x \partial y} = 0$ almost everywhere in $[0, 1]^2$. Note that W and M are singular copulas. An example of a non-singular copula is the independence copula Π defined by $\Pi(x, y) = xy$.

Obviously, the L_p-norm of a copula is at least 1. Moreover, observe that $V_C([x, x'] \times [y, 1]) = x' + C(x, y) - x - C(x', y) \geq 0$ implies $C(x', y) - C(x, y) \leq x' - x$, for any $y \in [0, 1]$ and $0 \leq x \leq x' \leq 1$, and similarly $C(x, y') - C(x, y) \leq y' - y$, for any $x \in [0, 1]$ and $0 \leq y \leq y' \leq 1$. Together with the increasingness of a copula, this implies that any copula is 1–1-Lipschitz. For any $p \in [1, \infty]$, there exists at least one copula that is 1-p-Lipschitz, the smallest one being $T_p^{\mathbf{Y}}$.

Our object of study in this paper are exactly these 1-p-Lipschitz copulas. We introduce the following notations:

(i) $\mathscr{C}$: the class of all copulas;
(ii) $\bar{\mathscr{C}_p}$: the class of 1-p-Lipschitz copulas;
(iii) $\mathscr{C}_p$: the class of copulas that are 1-p-Lipschitz, but not 1-p'-Lipschitz for any $p' > p$.

Clearly, $\bar{\mathscr{C}_p} = \cup_{r \geq p} \mathscr{C}_r$ and $\bar{\mathscr{C}_1} = \mathscr{C}$. Also, the family $(\bar{\mathscr{C}_p})_{p \in [1,\infty]}$ forms a decreasing inclusion chain w.r.t. p, while $(\mathscr{C}_p)_{p \in [1,\infty]}$ yields a partition of the class $\mathscr{C}$. Since the Yager triangular norms are copulas, and $T_p^{\mathbf{Y}} < T_{p'}^{\mathbf{Y}}$ for $p < p'$, it follows that $T_p^{\mathbf{Y}}$ is the smallest element of $\mathscr{C}_p$.

3. How to identify copulas belonging to $\bar{\mathscr{C}_p}$?

In order to determine whether a given copula belongs to one of the classes $\bar{\mathscr{C}_p}$, we need to be able to determine its L_p-norm and verify whether it equals 1. The following result is helpful for that purpose and is based on the fact that the Lipschitz continuity of a function is related to the boundedness of its first partial derivatives.

Proposition 3.1 (*[17,18]*). *A continuous real function $f : E \to \mathbb{R}$ is k-p-Lipschitz if and only if*

$$\left\| \left(\frac{\partial f(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{x})}{\partial x_n} \right) \right\|_q \leq k, \tag{4}$$