Nonlinear Analysis 72 (2010) 3759–3766

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Sub-shadowings

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ARTICLE INFO

Article history:
Received 23 April 2009
Accepted 12 January 2010

MSC:
primary 54H20
37B40
37D45
secondary 28D20

Keywords:
Ergodic pseudo-orbit
Shadowing
$\underline{d}$-shadowing
Chain transitive
Chain mixing

ABSTRACT

An extended concept of δ pseudo-orbit called δ-ergodic pseudo-orbit is considered. If any δ-ergodic pseudo-orbit of a system is shadowed by a point along a set of positive lower density then we show that this system is chain mixing and if it is minimal then it is topologically weakly mixing and so has Li–Yorke Chaos. A SFT has this property if and only if it is mixing.

1. Introduction

Suppose that X is a compact metric space and $f : X \to X$ a continuous and surjective map. A δ-*chain* from x to y of length n is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $d(f(x_i), x_{i+1}) \leq \delta$ for $i = 0, \ldots, n-1$. A δ-chain approximates an orbit and this approximation is usually for a short term. However, there are systems where this makes for long terms. Wherever this happens, one says that the δ-chain is shadowed by a point. The most eminent result in this regard is the so-called *Shadowing Lemma* due to [1].

A sequence $\xi = \{x_i\}_{i=0}^{\infty}$ of points is called δ *pseudo-orbit* if for any $i \geq 0$, $d(f(x_i), x_{i+1}) \leq \delta$. It is called δ-*ergodic pseudo-orbit* [2] if

$$\limsup \frac{|{B_n}^c(\xi, \delta)|}{n} = 0,$$

where ${B_n}^c(\xi, \delta) = \{i \in \{0, 1, \ldots, n\} : \ d(f(x_i), x_{i+1}) \geq \delta\}$. So a δ-ergodic pseudo-orbit may be presented as

$$x_0, x_1, \ldots, x_{m_1};\ x_{m_1+1}, \ldots, x_{m_2};\ x_{m_2+1}, \ldots$$

where $x_{m_i+1}, \ldots, x_{m_{i+1}}$ is a δ-chain of length $m_{i+1} - (m_i + 1) \geq 0$ and $M = \{m_i\}_{i \in \mathbb{N}}$ has density zero, $d(M) = \lim_{n\to\infty}(|M \cap \{0, \ldots, n\}|/n) = 0$. We say that a δ pseudo-orbit (resp. δ-ergodic pseudo-orbit) is ϵ-*shadowed* (resp. ϵ-*ergodic-shadowed* [2]) by a true orbit if there exists $z \in X$ such that for any $i \geq 0$ (resp. for any $i \in \Lambda$ with $d(\Lambda) = 1$), $d(f^i(z), x_i) \leq \epsilon$. The map f is said to have shadowing (resp. ergodic-shadowing [2]) property or briefly POTP if for any ϵ there exists δ such that any δ pseudo-orbit (resp. δ-ergodic pseudo-orbit) is ϵ-shadowed (resp. ϵ-ergodic-shadowed) by a true orbit.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2010.01.014

In *sub-shadowings* one considers a δ-ergodic pseudo-orbit and a shadowing along a large set of x_i's. We will see in the next section that if this large set has a subset of positive density, or equivalently, it has positive lower density then the system has some nice dynamical properties. For instance, it is *chain transitive* and in particular, it is *chain mixing*. A map f is called chain transitive if for any $\delta > 0$ and any two points x and y there exists a δ-chain from x to y. It is said to be *chain recurrent* if for any $\delta > 0$ and any x in X there is a δ-chain from x to itself. If for any $k \geq 1$, f^k is chain transitive then f is said to be *totally chain transitive*. A dynamical system (X, f) has chain mixing property if for any $\epsilon > 0$ there exists $N > 0$ such that for any x, y in X and any $n > N$ we have ϵ-chain of length n from x to y.

We recall some concepts. There are various kinds of *density* for natural sequences [3]. We already have defined the *density* of a sequence B as

$$d(B) = \lim_{n\to\infty} \frac{|B \cap \{1, \dots, n\}|}{n}$$

where $|A|$ denotes the cardinal number of A. *Lower density* and *upper density* are defined similarly with lim inf and lim sup, respectively. The *upper Banach density* of B is

$$d^*(B) = \limsup_{m-n\to\infty} \frac{|B \cap \{m, \dots, n\}|}{m - n + 1}.$$

A set $B \subseteq \mathbb{N}$ is *syndetic* if it has bounded gaps or equivalently a finite union of the shifts of B is equal to $\mathbb{N}$ [4,5]. That is for a natural number r, $\bigcup_{i=1}^{r} B - i = \mathbb{N}$. A sequence B is called *thick* set if it contains intervals of natural numbers of arbitrarily length [4,5]. That is

$$\forall n \in \mathbb{N}, \quad \exists\, x \in \mathbb{N}, \text{ s.t. } \{x, x+1, \dots, x+n\} \subset B.$$

It can be easily checked that B has full upper Banach density, $d^*(B) = 1$, if and only if it is thick. Also, any syndetic set has positive lower density.

The main concept that we introduce in this paper is $\underline{d}$-shadowing property which is the subject of Section 2. A system has this property if any δ-ergodic pseudo-orbit is traced along a set of positive lower density. We will investigate what properties a system with $\underline{d}$-shadowing property holds. Amongst these properties are totally chain transitivity (Theorem 2.5) and chain mixing (Corollary 2.6). Also we show that a minimal system with $\underline{d}$-shadowing property is weakly mixing (Theorem 2.8). In Section 3, we see that any system (Y, g) with a unique maximal attractor, X, so that $g|_X$ is exact, has $\underline{d}$-shadowing property (see Remark 3.2). In Section 4, we examine $\underline{d}$-shadowing property in subshifts. In particular, we show that any subshift of finite type is mixing iff it has $\underline{d}$-shadowing property (Theorem 4.3).

Lemma 1.1. *A set B of natural numbers has positive lower density if and only if $B \cap A \neq \emptyset$ for any sequence A with $\bar{d}(A) = 1$. In particular, $\bar{d}(A \cap B) > 0$.*

Proof. The first part is immediate by using the identity $\underline{d}(A) = 1 - \bar{d}(A^c)$, see also [6]. The second part follows from the fact that $1 = \bar{d}(A) \leq \bar{d}(A \cap B) + \bar{d}(A \setminus B)$ and the first part. □

2. $\underline{d}$-shadowing

In this section we introduce new concepts of shadowing and then we will investigate some basic dynamical properties of a system with these properties. First a lemma. Let

$$B_n(z, \xi, \epsilon) := \{i \in \{0, \dots, n\} : \ d(f^i(z), x_i) \leq \epsilon\}, \qquad B(\epsilon, \xi, z) := \cup_{n\geq 0} B_n(z, \xi, \epsilon).$$

Definition 2.1. Suppose that (X, f) is a topological dynamical system. If for any $\epsilon > 0$ there exists $\delta > 0$ such that any δ-ergodic pseudo-orbit ξ is ϵ-shadowed by a true orbit $\{f^i(z)\}_{i\in\mathbb{N}}$ in such a way that either

(1) $\underline{d}(B(\epsilon, \xi, z)) > 0$, or
(2) $\bar{d}(B(\epsilon, \xi, z)) > 1/2$

then we say that f has $\underline{d}$-*shadowing* property in case (1) and $\bar{d}$-*shadowing* in case (2).

Theorem 2.2. *Let $f : X \to X$ be a continuous map. If f has $\underline{d}$-shadowing property or $\bar{d}$-shadowing property then f is chain transitive.*

Proof. Let $\epsilon > 0$ and choose x, y in X. First suppose that f satisfies (1). Choose $M = \{m_i\}_{i=1}^{\infty}$ an increasing subsequence of natural numbers such that $d(M) = 0$ and so that if $M_1 = \{1, 2, \dots, m_1\} \cup \{m_2 + 1, m_2 + 2, \dots, m_3\} \cup \dots \cup \{m_{2i} + 1, m_{2i} +$