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The existence of viable trajectories in state-dependent impulsive systems

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ABSTRACT

Suitable topological tools for studying the existence of viable and periodic viable trajectories in state-dependent impulsive differential equations and inclusions are provided. Some results based on the Ważewski retract method are applied. The techniques presented in the paper are illustrated with several examples.

1. Introduction

A lot of modern mathematical models in engineering, biology, demography, etc., have been described as impulsive differential equations. Nowadays, the literature concerning the subject is wide and too rich to be quoted in this note. Indeed, several monographs have appeared (see, e.g., [1–3]). An essential development of this area of research have been made in last two decades. Generally speaking, an impulsive differential equation (or, more generally, differential inclusion) consists of two elements: a differential equation (or inclusion) determining the dynamics between points (times or states) of impulses, and an impulse map that describes the set where impulses occur and the way the impulses act. If the resetting events are defined by a prescribed sequence of times, independent of the system state, we say that a time-dependent differential problem is given. We are interested in an alternative case where impulses are defined on a prescribed subset of the state space, so we are given *state-dependent impulsive differential equations* and inclusions; see, e.g., [4]. Several essential problems are studied in this subject: the suitable notion of a dynamical system that is generated by impulsive differential equations, its regularity (especially, regularity of the *resetting time* function) and stability, periodic and stationary trajectories, and many other analogs to ODEs or differential inclusions (see, e.g., [4–6]).

Our attention focuses on impulsive problems with constraints. For a given closed subset K of a state space, we look for trajectories remaining in K forever. They are called *viable* in K, where the notion has an obvious biological meaning that

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some species survives only if it satisfies suitable constraints, that is, if it does not leave a region of safety K. The existence of viable trajectories is an important first step in searching viable periodic or stationary solutions in the set of constraints. This problem is even more important when it relates to a multivalued setting which arises when a control problem is replaced by a suitable differential inclusion. Then, if we look for viable trajectories for the inclusion, we ask for controls allowing it to remain in K and survive. There are only a few papers concerning the *viability* in state-dependent impulsive systems in the sense that the whole set K is required to be a set of initial points of trajectories viable in K (see, e.g., [7]). Obviously, some tangency conditions are required on the set K.

If we do not assume so restrictive tangency conditions, we allow trajectories for a non-impulsive problem, even all of them, to leave K. They escape from K through a so-called *exit set*. To prevent this, we place a barrier M, being an essential subset of the exit set, and define an impulse map moving trajectories back to the set K. The problem we deal with in the present paper is: what are sufficient conditions for a non-impulsive differential equation (or inclusion), a set K, the exit set, the barrier M and the impulsive map to obtain viable trajectories for an impulsive system. The aim is to provide suitable topological tools detecting such trajectories. We show a connection between the piecewise continuous dynamics of an impulsive system with topological invariants for flows, and a discrete dynamical (multivalued) system on the exit set with an appropriate index theory for multivalued maps. As we shall notice, topological properties of the exit set are important to solve the problem. It is strictly related to the celebrated Ważewski retract method and the fixed point theory.

The methods presented in the paper are followed by several examples. They are chosen to be very simple to illustrate the techniques in a transparent way. It is not our goal to prove new fixed point results. The statements contained in the text are often given without proof, being consequences of suitable fixed point or invariant set theorems. The key point was to show the way in which these theorems could be successfully applied. In Sections 2 and 3 we present two approaches, namely, the fixed point index and the Conley index approach, for both single-valued and multivalued maps. In Section 3 we show that a multivalued setting is adequate for our problem of the existence of viable trajectories, even for single-valued differential equations. Section 4 describes how the Poincaré section technique together with the additivity property of the fixed point index can be used to detect nontrivial periodic solutions in impulsive differential equations.

2. Fixed point index approach

2.1. Impulsive semidynamical systems

Let $K \subset \mathbb{R}^n$ be an arbitrary closed subset. We are interested in solving the problem

$$\text{Does there exist a viable in } K \text{ solution to the equation } \dot{x} = f(x)? \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz with sublinear growth, i.e., $|f(x)| \le c(1 + |x|)$ for some constant $c \ge 0$ and every $x \in \mathbb{R}^n$. By a *solution* we mean an a.e. continuous map $x : [0, \infty) \to \mathbb{R}^n$ which is a.e. differentiable, $\dot{x}(t) = f(x(t))$ for a.e. $t \ge 0$, and discontinuity is met only in times when the solution attains some block contained in the exit set $K^-(f) := \{x_0 \in K \mid x(0) = x_0 \text{ and } \forall \varepsilon > 0 : x((0, \varepsilon)) \not\subset K\}$. A *viable* solution is one that satisfies $x(t) \in K$ for every $t \ge 0$.

Assume that

$$K^-(f) \text{ is closed.} \tag{2}$$

As mentioned in the Introduction, problem (1) has an affirmative answer if $K^-(f)$ is not a strong deformation retract of K (see [8]). Obviously, the solution we get is smooth since it does not touch the boundary at all. In what follows we are interested in the case where

$$\text{all absolutely continuous solutions to } \dot{x} = f(x) \text{ leave the set } K. \tag{3}$$

In consequence, K is continuously strongly deformed onto $K^-(f)$. Namely, if we denote by $\pi(\cdot, \cdot)$ a dynamical system generated by f, then the exit function $\tau_K : K \to [0, \infty)$, $\tau_K(x_0) := \sup\{t \ge 0 \mid \pi(\{x_0\} \times [0, t])\} \subset K$ is continuous, and we can define $h : K \times [0, 1] \to K$, $h(x_0, \lambda) := \pi(x_0, \lambda\tau_K(x_0))$. Putting $r(x_0) := h(x_0, 1)$, we obtain a continuous map $r : K \to K^-(f)$ being a retraction.

Let $M \subset K^-(f)$ be an *impulse set*, i.e., there is an *impulse map* $I : M \to K$ thrusting some running away solutions back to K. For a while we assume that I is a single-valued continuous map with a compact image satisfying $I(M) \cap M = \emptyset$. We define

$$g : M \to K^-(f), \qquad g(x) := r(I(x)) \quad \text{for } x \in M. \tag{4}$$

Consider a hybrid state-dependent system

$$\begin{cases} \dot{x}(t) = f(x(t)) & \text{for a.e. } t \ge 0, \\ x(t^+) := \lim\limits_{s \to t^+} x(s) = I(x(t)) & \text{for } x(t) \in M. \end{cases} \tag{5}$$

It generates an *impulsive semidynamical system* (see [4]). One also can check that it generates a left-continuous dynamical system in the sense of [5] only if M is a connected component of $K^-(f)$. Indeed, since $I(M)$ is compact and M is closed, for every $x_0 \in K \setminus M$ the sequence $\sigma_0(x_0) = 0$, $\sigma_1(x_0) = \phi(x_0)$, $\sigma_{m+1}(x_0) = \sigma_m(x_0) + \phi(I(g^{m-1}(r(x_0))))$ satisfies the condition $\lim_{m\to\infty} \sigma_m(x_0) = \infty$, where $\phi(x) = \inf\{s > 0 \mid \pi(\{x\} \times [0, s)) \cap M = \emptyset \text{ and } \pi(x, s) \in M\}$. Moreover, the function $\phi : \pi^{-1}(K^-(f)) \to [0, \infty]$ is then continuous outside M, and hence, all assumptions required for a left-continuous