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Differential mixed variational inequalities in finite dimensional spaces☆

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ABSTRACT

In this paper, we introduce and study a class of differential mixed variational inequalities in finite dimensional Euclidean spaces. Under various conditions, we obtain linear growth and bounded linear growth of the solution set for the mixed variational inequalities. Moreover, we present some conclusions which enrich the literature on the mixed variational inequalities and generalize the corresponding results of [4]. In particular we prove existence theorems for weak solutions of a differential mixed variational inequality in the weak sense of Carathéodory by using a result on differential inclusions involving an upper semicontinuous set-valued map with closed convex values. Also by employing the results from differential inclusions we establish a convergence result on Euler time-dependent procedure for solving initial-value differential mixed variational inequalities.

1. Introduction

Ordinary differential equations (ODEs) with smooth right-hand functions arise naturally in applied mathematics. Differential inclusions (DIs) constitute a branch of the general theory of differential equation now under active development. They arise in models for many dynamical processes and they also provide a powerful tool for various branches of mathematical analysis. The great impetus to study DIs came from the development of optimal control theory, i.e., from a control system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad u \in U,$$

where u is a control parameter. DIs, which are used to derive sufficient conditions for optimality play an essential role in the theory of control under conditions of uncertainty and in differential game theory (see [1–4]).

Variational inequalities (VI) also arise naturally and have applications in engineering and economics. Lescarret [5] and Browder [6] originally considered a mixed variational inequality (MVI) in connection with its numerous applications in mathematical physics (see also [7,8]). Let K be a nonempty closed convex set of R^m. Let $S : R^m \to R^m$ be a given function and $\varphi : R^m \to (-\infty, +\infty]$ be a proper lower semicontinuous (l.s.c.) convex functional. An MVI(K, S, φ) problem is to find an element $u \in K$ such that

$$\langle S(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K. \tag{1.1}$$

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It is easy to see that MVI (1.1) is a generalization of the classical VI. In fact, if φ is the indicator functional on K, that is,

$$\varphi(u)=\begin{cases}0, & \text{if } u\in K;\\ +\infty, & \text{if } u\notin K,\end{cases}$$

then MVI (1.1) reduces to the following classical VI: find a vector $u\in K$ such that

$$\langle S(u), v-u\rangle \geq 0,\quad \forall v\in K. \tag{1.2}$$

In this paper, we denote by $\mathrm{SOL}(K,S,\varphi)$ and $\mathrm{SOL}(K,S)$ the solution sets of MVI (1.1) and VI (1.2) respectively.

We would like to point out that if K is a nonempty, closed and convex cone of R^m, then Theorem 2.1(i) of [9] shows that MVI (1.1) reduces to the following φ-complementarity problem (φ-CP): find a vector $u\in K$ such that

$$\langle S(u), u\rangle+\varphi(u)=0,\qquad \langle S(u), v\rangle+\varphi(v)=0,\quad \forall v\in K.$$

For more details on the connection between MVIs and φ-CPs, we refer to [9] and the references therein.

Linear complementarity systems (see, for example, [10–12]) belong to the more general class of differential variational inequalities (DVIs). Recently Pang and Stewart [4] put forward the concept of differential variational inequality (DVI), which in fact, is the terminology of variational inequality of evolution (VIE) used in Aubin and Cellina [1]. A DVI comprises of two major components: an ODE and a VI. Let $f: R^{1+n+m}\to R^n$ and $S: R^{1+n+m}\to R^m$ be two continuous vector functions. Let K be a nonempty closed convex subset of R^m and let $\Gamma: R^{2n}\to R^n$ be a bounded function and $T>0$ a terminal time. The DVI defined by the triplet of functions (f,S,Γ), the set K, and the scalar T is to find time-dependent trajectories $x(t)$ and $u(t)$ that satisfy condition (1.3) in the weak sense of Carathéodory for $t\in[0,T]$,

$$\begin{cases}\dot{x}(t)=f(t,x(t),u(t))\\ u(t)\in \mathrm{SOL}(K,S(t,x(t),\cdot))\end{cases} \tag{1.3}$$

and also the algebraic condition

$$\Gamma(x(0),x(T))=0. \tag{1.4}$$

Pang and Stewart [4] introduced and studied the following initial-value DVI:

$$\begin{cases}\dot{x}(t)=f(t,x(t))+B(t,x(t))u(t)\\ u(t)\in \mathrm{SOL}(K,G(t,x(t))+S)\\ x(0)=x_0,\end{cases} \tag{1.5}$$

where K is a nonempty closed convex subset of R^m, $\Omega\equiv[0,T]\times R^n$, $(f,B,G):\Omega\to R^n\times R^{n\times m}\times R^m$, and $S: R^m\to R^m$ are given functions. Under various conditions Pang and Stewart [4] obtained the existence of a Carathéodory weak solution for initial-value DVIs. By using some results concerned with DIs, they established the convergence of the Euler time-stepping procedure for solving initial-value DVIs. Furthermore they studied the solution dependence on initial conditions in DVIs and obtained some interesting results. For some related results, we refer to Han and Pang [13] and Pang and Shen [14].

Very recently, Stewart [15] investigated the uniqueness for solutions of differential complementarity problems (DCPs) and gave some interesting characterization results concerned with the uniqueness for solutions of DCPs, which have implications for a range of different formulations of dynamic systems with complementarity constraints.

On the other hand, since DVIs have many important applications in analytic mechanics, differential Nash games and other fields, several authors have developed numerical schemes of DVIs with continuous/discontinuous state trajectories by using monotone techniques (see, for example, [16,17]).

Motivated and inspired by the work mentioned above, in this paper we introduce and study an initial-value differential mixed variational inequality (DMVI) with the right-hand function in the ODE being affine in u and with the MVI function being separable in u:

$$\begin{cases}\dot{x}(t)=f(t,x(t))+B(t,x(t))u(t)\\ u(t)\in \mathrm{SOL}(K,G(t,x(t))+S,\varphi)\\ x(0)=x_0,\end{cases} \tag{1.6}$$

where K is a nonempty closed convex subset of R^m, $\Omega\equiv[0,T]\times R^n$, $(f,B,G):\Omega\to R^n\times R^{n\times m}\times R^m$, $S: R^m\to R^m$ are given functions, and $\varphi: R^m\to(-\infty,+\infty]$ is a proper l.s.c. convex functional. It is easy to see that, if φ is the indicator functional on subset K, then the problem (1.6) reduces to problem (1.5).

We are interested in finding time-dependent trajectories $x(t)$ and $u(t)$ such that (1.6) holds in the weak sense of Carathéodory for $t\in[0,T]$. This means that x is an absolutely continuous function on $[0,T]$, that u is an integrable function on $[0,T]$, and that the differential equation need only be satisfied for almost all t; moreover, the membership for $u(t)$ means that for any continuous function $\tilde{u}:[0,T]\to K$,

$$\int_0^T\{\langle G(t,x(t))+S(u(t)), \tilde{u}(t)-u(t)\rangle+\varphi(\tilde{u}(t))-\varphi(u(t))\}\mathrm{d}t\geq 0, \tag{1.7}$$