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Asymptotic behavior of a class of non-autonomous degenerate parabolic equations

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ABSTRACT

We investigate the asymptotic behavior of solutions of a class of non-autonomous degenerate parabolic equations on $\mathbb{R}^n$ with unbounded external forcing terms. The existence of a pullback global attractor is proved in $L^2(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n) \bigcap W^{1,r}(\mathbb{R}^n)$ for any $p \geq r \geq 2$.

1. Introduction

This paper is concerned with the long-time behavior of the following non-autonomous degenerate parabolic equations defined on the entire space $\mathbb{R}^n$:

$$u_t - \operatorname{div}\left(|\nabla u|^{r-2}\, \nabla u\right) + \lambda u + f(x, u) = g(x, t), \quad x \in \mathbb{R}^n,\ t > \tau \text{ and } \tau \in \mathbb{R}, \tag{1.1}$$

where $\lambda > 0$ and $r \geq 2$ are constants, g is an external forcing term given in $L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$, and f is a nonlinear function satisfying some dissipative conditions.

In the autonomous case, global attractors of Eq. (1.1) have been studied by several authors in [1–7] and the references therein. For the non-autonomous equation, the existence of uniform attractors of (1.1) has been proved in [8] under the conditions that the domain is *bounded* and the external term g is *uniformly bounded* in L^s with $s \geq 2$. In the present paper, we will prove the existence of pullback attractors for the non-autonomous equation (1.1) defined on the *unbounded* domain $\mathbb{R}^n$ with *unbounded* g in L^2. For other types of non-autonomous equation, the attractors have been investigated by many people in [9–30]. In particular, when PDEs are defined on unbounded domains, such attractors have been examined in [9,25,26] for almost periodic external terms and in [12,13,29,31,30] for unbounded external terms. The purpose of this paper is to study the non-autonomous attractors for the degenerate parabolic equation (1.1) with unbounded external terms defined on $\mathbb{R}^n$.

The main difficulty for proving existence of attractors for PDEs defined on $\mathbb{R}^n$ is of course the non-compactness of Sobolev embeddings caused by the unboundedness of the domain. One way to overcome this difficulty is the energy equation approach introduced by Ball in [32,33] and used in [12,13,34–36,25,37–40]. In the present paper, we will resolve this difficulty by using the tail-estimates method introduced in [41]. More precisely, we will show that the solutions of the

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non-autonomous equation (1.1) are uniformly small in a sense for large space and time variables. This fact enables us to prove the pullback asymptotic compactness of solutions in $L^2(\mathbb{R}^n)$ and hence the existence of pullback attractors in the same space. Next, we show that the attractor in $L^2(\mathbb{R}^n)$ is actually a pullback attractor in $L^p(\mathbb{R}^n)$ for any $p \geq r$. To this end, we will first prove a necessary and sufficient condition for convergence of a sequence in $L^p(\mathbb{R}^n)$, which is motivated by the work of [42]. By applying the convergence criteria and uniform estimates in $L^p(\mathbb{R}^n)$, we will then be able to prove the existence of a pullback attractor for (1.1) in $L^p(\mathbb{R}^n)$ which actually coincides with the attractor in $L^2(\mathbb{R}^n)$. Finally, we show that the pullback attractor in $L^2(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n)$ is also a pullback attractor for Eq. (1.1) in $W^{1,r}(\mathbb{R}^n)$.

The outline of this paper is as follows. The next section gives a brief review of pullback attractor theory for non-autonomous systems. In Section 3, we provide a necessary and sufficient condition for convergence of a sequence in L^p. Section 4 is concerned with the existence of a cocycle for the non-autonomous equation (1.1) on $\mathbb{R}^n$. In Section 5 we derive uniform estimates of solutions for large space and time variables. In the last section, we prove the existence of a pullback global attractor in $L^2(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n) \bigcap W^{1,r}(\mathbb{R}^n)$.

The following notations will be used throughout the paper. We denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and inner product in $L^2(\mathbb{R}^n)$ and use $\| \cdot \|_p$ to denote the norm in $L^p(\mathbb{R}^n)$. Otherwise, the norm of a general Banach space X is written as $\| \cdot \|_X$. The letters C and C_i $(i = 1, 2, \dots)$ are generic positive constants which may change their values from line to line or even in the same line.

2. Preliminaries

This section is devoted to a brief review of the pullback attractor theory for non-autonomous dynamical systems. The reader is referred to [12,13,17,43,44] for more details. In the following, Ω is a nonempty set, X is a metric space with distance $d(\cdot, \cdot)$, and $\mathcal{D}$ is a collection of families of subsets of X:

$$\mathcal{D} = \{D = \{D(\omega)\}_{\omega \in \Omega} : D(\omega) \subseteq X \text{ for every } \omega \in \Omega\}.$$

Definition 2.1. A family of mappings $\{\theta_t\}_{t \in \mathbb{R}}$ from Ω to itself is called a family of shift operators on Ω if $\{\theta_t\}_{t \in \mathbb{R}}$ satisfies the group properties:

(i) $\theta_0 \omega = \omega, \forall \omega \in \Omega$;
(ii) $\theta_t(\theta_\tau \omega) = \theta_{t+\tau}\omega, \forall \omega \in \Omega$ and $t, \tau \in \mathbb{R}$.

Definition 2.2. Let $\{\theta_t\}_{t \in \mathbb{R}}$ be a family of shift operators on Ω. Then a continuous θ-cocycle ϕ on X is a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \qquad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

which satisfies, for all $\omega \in \Omega$ and $t, \tau \in \mathbb{R}^+$,

(i) $\phi(0, \omega, \cdot)$ is the identity on X;
(ii) $\phi(t + \tau, \omega, \cdot) = \phi(t, \theta_\tau \omega, \cdot) \circ \phi(\tau, \omega, \cdot)$;
(iii) $\phi(t, \omega, \cdot) : X \to X$ is continuous.

Definition 2.3. Let $\mathcal{D}$ be a collection of families of subsets of X. Then $\mathcal{D}$ is called inclusion-closed if $D = \{D(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and $\tilde{D} = \{\tilde{D}(\omega) \subseteq X : \omega \in \Omega\}$ with $\tilde{D}(\omega) \subseteq D(\omega)$ for all $\omega \in \Omega$ imply that $\tilde{D} \in \mathcal{D}$.

Definition 2.4. Let $\mathcal{D}$ be a collection of families of subsets of X and $\{K(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega \in \Omega}$ is called a pullback absorbing set for ϕ in $\mathcal{D}$ if, for every $B \in \mathcal{D}$ and $\omega \in \Omega$, there exists $T(\omega, B) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \geq T(\omega, B).$$

Definition 2.5. Let $\mathcal{D}$ be a collection of families of subsets of X. Then ϕ is said to be $\mathcal{D}$-pullback asymptotically compact in X if, for every $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$.

Definition 2.6. Let $\mathcal{D}$ be a collection of families of subsets of X and $\{\mathcal{A}(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. Then $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ is called a $\mathcal{D}$-pullback global attractor for ϕ if the following conditions are satisfied, for every $\omega \in \Omega$:

(i) $\mathcal{A}(\omega)$ is compact in X;
(ii) $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ is invariant; that is,

$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t \omega), \quad \forall t \geq 0;$$

(iii) $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ attracts every set in $\mathcal{D}$; that is, for every $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$,

$$\lim_{t \to \infty} d(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d is the Hausdorff semi-metric.