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The state constrained bilateral minimal time function

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Abstract

We generalize, to the bilateral case (that is, with variable initial and end points), the main results of Nour and Stern [C. Nour, R.J. Stern, Regularity of the state constrained minimal time function, Nonlinear Anal. 66 (1) (2007) 62–72] and Stern [R.J. Stern, Characterization of the state constrained minimal time function, SIAM J. Control Optim. 43 (2004) 697–707], where the regularity and Hamilton–Jacobi characterization of the state constrained (unilateral) minimal time function were studied.
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1. Introduction

We consider a state constrained control system governed by a differential inclusion. We are given a multifunction F mapping $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$, and a closed set $S \subset \mathbb{R}^n$. Associated with F is the following S-state constrained differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(x(t)) \quad \text{a.e. } t \in [0, T], \\ x(t) \in S \quad \forall t \in [0, T]. \end{cases} \tag{1}$$

A *solution* $x(\cdot)$ of (1) is taken to mean an absolutely continuous function $x : [0, T] \longrightarrow \mathbb{R}^n$ which, together with $\dot{x}$, its derivative with respect to t, satisfies the two inclusions of (1). We refer to such an x as an *S-trajectory* (of F).

The *S-constrained bilateral minimal time function* $T_S : S \times S \longrightarrow [0, +\infty]$ is defined as follows: For $(\alpha, \beta) \in S \times S$, $T_S(\alpha, \beta)$ is the minimum time taken by an S-trajectory to go from α to β. (When no such S-trajectory exists, $T_S(\alpha, \beta)$ is taken to be $+\infty$.) In the unconstrained case (when $S = \mathbb{R}^n$), the function $T_S(\cdot, \cdot)$ coincides with the bilateral minimal time function (denoted by $T(\cdot, \cdot)$) defined in [8], where Clarke and Nour studied the Hamilton–Jacobi equation of minimal time control in a domain which contains the target set, and utilized the function $T(\cdot, \cdot)$ for the construction of solutions [8, Section 5], and for the study of the existence of time-semigeodesic trajectories [8, Section 6]; see also Nour [16]. The function $T(\cdot, \cdot)$ was studied in depth in Nour [15], where the regularity conditions and a characterization of $T(\cdot, \cdot)$ as the unique proximal solution of partial Hamilton–Jacobi

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equations were provided. These results extended to the bilateral case ones known for the (unilateral) minimal time function, which has an extensive literature. In this regard, see for example [4–6,14,18,19,21,22,25,26] for the regularity study of this function, and [1–3,8,11,23,26] for its Hamilton–Jacobi characterization. The reader can refer to Bardi and Capuzzo-Dolcetta [2, Chapter 4] for a thorough history of such results.

The regularity and Hamilton–Jacobi characterization of the state constrained (unilateral) minimal time function was studied (apparently for the first time) in Nour and Stern [17] and Stern [24], respectively. In [17], the Lipschitz continuity of this function was studied, as was its link with the unconstrained case, while [24] provides its characterization as the unique proximal solution of Hamilton–Jacobi equations with certain boundary conditions. The goal of the present article is to generalize these results to the bilateral case.

The plan of the paper is as follows: In the next section we present basic definitions and hypotheses. Then in Section 3 we shall study the regularity of the state constrained bilateral minimal time function $T_S(\cdot,\cdot)$. Section 4 is devoted to its Hamilton–Jacobi characterization.

2. Definitions and hypotheses

We denote by $\|\cdot\|$ the Euclidean norm and by $\langle,\rangle$ the usual inner product. For $\rho > 0$ and $\alpha \in \mathbb{R}^n$ we denote

$$B(\alpha;\rho) := \{x \in \mathbb{R}^n : \|x-\alpha\| < \rho\} \quad \text{and} \quad \bar{B}(\alpha;\rho) := \{x \in \mathbb{R}^n : \|x-\alpha\| \le \rho\}.$$

The open (resp. closed) unit ball in $\mathbb{R}^n$ is denoted by B (resp. $\bar{B}$). For a set $A \subset \mathbb{R}^n$, comp A, int A, bdry A and co A are the complement (with respect to $\mathbb{R}^n$), the interior, boundary and convex hull of A, respectively. The distance from a point x to a set A is denoted by $d_A(x)$. Given a set A and a point x, we denote by $\mathrm{proj}_A(x)$ the set of closest points of x onto A, that is, the set of points $\alpha \in A$ which satisfy $d_A(x) = \|\alpha - x\|$.

Now we provide certain geometric definitions from nonsmooth analysis. Our general reference for such constructs is the book of Clarke, Ledyaev, Stern and Wolenski [7]. For a nonempty closed subset A of $\mathbb{R}^n$ and a point $\alpha \in A$, the *Bouligand tangent cone*, *proximal normal cone*, *limiting normal cone* and *Clarke normal cone* to A at α are defined respectively by:

- $\mathrm{Tan}_A^B(\alpha) := \{\lim \frac{\alpha_i-\alpha}{t} : \alpha_i \xrightarrow{A} \alpha, t_i \downarrow 0\}$;
- $N_A^P(\alpha) := \{t(x-\alpha) : \alpha \in \mathrm{proj}_A(x), t \ge 0\}$;
- $N_A^L(\alpha) := \{\lim \zeta_i : \zeta_i \in N_A^P(\alpha_i), \alpha_i \xrightarrow{A} \alpha\}$;
- $N_A^C(\alpha) := \mathrm{co}\, N_A^L(\alpha)$.

Here $\alpha_i \xrightarrow{A} \alpha$ means that $\alpha_i \in A$ and $\alpha_i \longrightarrow \alpha$.

Now let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let x be a point in its effective domain, that is, $x \in \mathrm{dom}\, f := \{x' : f(x') < +\infty\}$. A vector $\zeta \in \mathbb{R}^n$ is said to be a *proximal subgradient* of f at x if $(\zeta,-1) \in N^P_{\mathrm{epi}\, f}(x, f(x))$, where epi f the epigraph of f is defined by $\mathrm{epi}\, f := \{(x',r) : r \ge f(x')\}$. The set of all such ζ is denoted by $\partial_P f(x)$ and is referred to as the *proximal subdifferential*. We can prove that one has $\zeta \in \partial_P f(x)$ if and only if there exists $\sigma = \sigma(x,\zeta) \ge 0$ such that

$$f(y) - f(x) + \sigma\|y-x\|^2 \ge \langle \zeta, y-x\rangle,$$

for all y in a neighborhood of x. (The latter condition is known as the *proximal subgradient inequality*.)

Now let $S \subset \mathbb{R}^n$ be a closed set and let $F : R^n \rightrightarrows \mathbb{R}^n$ be a multifunction. As mentioned in the introduction, the S-constrained bilateral minimal time function, denoted by $T_S(\cdot,\cdot)$, is defined by:

$$T_S(\alpha,\beta) := \inf\{T \ge 0 : x(t) \text{ is an } S\text{-trajectory of } F \text{ with } x(0)=\alpha \text{ and } x(T)=\beta\}.$$

If no S-trajectory exists between α and β, then $T_S(\alpha,\beta)$ is taken to be $+\infty$.

We shall assume throughout that F satisfies the following conditions:

- For every $x \in \mathbb{R}^n$, $F(x)$ is a nonempty compact convex set.
- The *linear growth condition*: For some positive constants γ and c, and for all $x \in \mathbb{R}^n$,

$$v \in F(x) \Longrightarrow \|v\| \le \gamma\|x\| + c.$$

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