Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 69 (2008) 3610–3628
Nonlinear Analysis
www.elsevier.com/locate/na

Bifurcation of periodic orbits in a class of planar Filippov systems☆

Zhengdong Du[a,*], Yurong Li[b], Weinian Zhang[a]

[a] Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China
[b] School of Information Engineering, Southwestern University of Finance and Economics, Chengdu, Sichuan 610074, PR China

Received 29 August 2007; accepted 28 September 2007

Abstract

In this paper we discuss the perturbations of a general planar Filippov system with exactly one switching line. When the system has a limit cycle, we give a condition for its persistence; when the system has an annulus of periodic orbits, we give a condition under which limit cycles are bifurcated from the annulus. We also further discuss the stability and bifurcations of a nonhyperbolic limit cycle. When the system has an annulus of periodic orbits, we show via an example how the number of limit cycles bifurcated from the annulus is affected by the switching.

MSC: 34A36; 34C25; 37G15; 37N35

Keywords: Filippov system; Non-smooth system; Periodic orbit; Poincaré map; Bifurcation

1. Introduction

The study of bifurcation phenomena in *non-smooth dynamical systems* has become very active in recent years because non-smooth dynamical systems have been modelled in many fields. For example, relay feedback systems in control theory [2,3,6], switching circuits in power electronics [4], impact and dry frictions in mechanical engineering [8,13,14,24,29], a car brake system [36], etc. According to Leine and van Campen [25], there are basically three types of non-smooth dynamical systems, namely *non-smooth continuous systems*, *Filippov systems* and *systems which expose discontinuities in time of the state*. Due to the variety of the non-smoothness, these systems exhibit not only standard bifurcations, but also complicated nonstandard bifurcation phenomena not existing in smooth systems, such as grazing [5,10], border-collision [30] and sliding effects [8], etc. The study and classification of various kinds of nonstandard bifurcation phenomena for non-smooth systems have attracted great attentions during the last decade, see, for example, [5–8,10,13,14,18,19,25,30] and the references therein.

Another important issue is to investigate if some well-known bifurcations occurring in smooth systems, such as Hopf bifurcation, homoclinic bifurcation and subharmonic bifurcation, also exist in non-smooth systems and if those

☆ Supported by the MOE Research Foundation for Returned Overseas Chinese Scholars, the NSFC (China) 10471101 and the SRFDP Grants 20050610003.
* Corresponding author. Tel.: +86 028 89539840.
E-mail address: zdu85@yahoo.com (Z. Du).

0362-546X/$ - see front matter
doi:10.1016/j.na.2007.09.045

known bifurcation methods can be generalized to non-smooth cases. In [26] Leine and Nijmeijer considered a non-smooth Hopf bifurcation from which an equilibrium hits the line of discontinuity and loses its stability while periodic orbits arise. In [35,36] Zou et al. discussed another case of non-smooth Hopf bifurcation for planar Filippov systems where the equilibrium always stays on the smooth line of discontinuity but the bifurcated periodic orbit crosses the line at least twice transversally. The focus-center-limit cycle bifurcation for a symmetric 3-dimensional piecewise linear system was discussed by Freire et al. [17]. In addition, Melnikov methods for homoclinic and subharmonic bifurcations were extended to non-smooth systems in [15,16,23,27]. As pointed out in [24], almost always the generalization of the bifurcation methods for smooth systems to non-smooth cases is a non-trivial task.

The purpose of this paper is to study the limit cycles bifurcated from periodic orbits of a planar Filippov system. Being an important class of non-smooth systems, a *Filippov system* [20] is a type of *piecewise smooth* (PWS) systems. The phase space of a PWS system is split into disjoint connected subregions such that the defining vector field is smooth in each subregion. Usually, the boundaries between the different regions are referred to as *switching manifolds* or *discontinuity surfaces*. Filippov systems have widespread applications, particularly in control theory. Various relay feedback systems can be classified as Filippov systems. There have been many papers (see e.g. in [2,3,21,22,28,33]) which dealt with the existence and stability of limit cycles for different kinds of linear relay feedback systems. Since solutions of a linear relay feedback system can be given explicitly in each subregion, conditions for the existence and stability of limit cycles can be obtained analytically by the method of state-space representations and the analysis of Poincaré maps. However, the problem of the existence and stability of limit cycles is much more difficult for nonlinear Filippov systems.

What we consider in this paper can be regarded as an extension of Poincaré bifurcation for smooth systems to planar Filippov systems. Assume that the state space $\mathbb{R}^2$ is split into two disjoint regions Σ_+ and Σ_- by the switching line $\Sigma_0 := \{x \in \mathbb{R}^2 : c^{\mathrm{T}}x = 0\}$ such that $\mathbb{R}^2 = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$, where $c = (c_1, c_2)^{\mathrm{T}} \in \mathbb{R}^2$ is a nonzero constant vector, $\Sigma_+ := \{x \in \mathbb{R}^2 : c^{\mathrm{T}}x > 0\}$ and $\Sigma_- := \{x \in \mathbb{R}^2 : c^{\mathrm{T}}x < 0\}$. Consider the following planar Filippov system

$$\dot{x} = g_+(x) + \varepsilon f_+(x), \quad x \in \Sigma_+, \tag{1.1a}$$

$$\dot{x} = g_-(x) + \varepsilon f_-(x), \quad x \in \Sigma_-, \tag{1.1b}$$

where $g_\pm, f_\pm \in C^2(\Sigma_\pm \cup \Sigma_0, \mathbb{R}^2)$ and $|\varepsilon| \le \varepsilon_0 \ll 1$ for some $\varepsilon_0 > 0$. Moreover, let the following conditions hold:

(H1) For $\varepsilon = 0$, the phase portrait of the unperturbed system of (1.1), i.e.,

$$\dot{x} = g_+(x), \quad x \in \Sigma_+, \tag{1.2a}$$

$$\dot{x} = g_-(x), \quad x \in \Sigma_-, \tag{1.2b}$$

has either a limit cycle Γ, or a periodic annulus $\mathcal{A}$ consisting of a one-parameter family of periodic orbits Γ_h, where $h \in J := (h_1, h_2) \subseteq \mathbb{R}$ and J is a nonempty interval.

(H2) The limit cycle Γ (or each Γ_h in $\mathcal{A}$) is a *transversal unimodal periodic orbit* of (1.2), i.e., it has only transversal intersections with Σ_0 and it intersects Σ_0 exactly twice.

(H3) The limit cycle Γ (or each Γ_h in $\mathcal{A}$) crosses Σ_0 counterclockwise with travelling time T^+, T^- (or T_h^+, T_h^-) in Σ_+, Σ_- respectively.

Here the assumption (H3) is not essential because if Γ (or each Γ_h in $\mathcal{A}$) crosses Σ_0 clockwise, one can reverse the time to satisfy (H3). A transversal unimodal periodic orbit of (1.2) that crosses Σ_0 counterclockwise is shown in Fig. 1.

In this paper, when (1.2) has a limit cycle Γ, we give a condition for its persistence; when (1.2) has a periodic annulus $\mathcal{A}$, we give a condition under which limit cycles are bifurcated from $\mathcal{A}$. We also discuss the stability and bifurcations of a nonhyperbolic limit cycle further. Moreover, when (1.2) has a periodic annulus $\mathcal{A}$, we show via an example how the number of limit cycles bifurcated from the annulus is affected by the switching.

This paper is organized as follows. The main results are presented in Section 2. We give estimates to the Poincaré maps of periodic orbits in Section 3 and the proofs of the main results in Section 4. In Section 5, we consider the stability and bifurcation of nonhyperbolic limit cycles. The discussions on the effects of switching and the relations to a smooth system are presented in Section 6. Finally we apply our results to a nonlinear Filippov system of Liénard type in Section 7.