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Endpoints of set-valued dynamical systems of asymptotic contractions of Meir–Keeler type and strict contractions in uniform spaces

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Abstract

In this paper, we introduce the concepts of the set-valued dynamical systems of asymptotic contractions of Meir–Keeler type and set-valued dynamical systems of strict contractions in uniform spaces and we present a method which is useful for establishing conditions guaranteeing the existence and uniqueness of endpoints of these contractions and the convergence to these endpoints of all generalized sequences of iterations of these contractions. The result, concerning the investigations of problems of the set-valued asymptotic fixed point theory, include some well-known results of Meir and Keeler, Kirk and Suzuki concerning the asymptotic fixed point theory of single-valued maps in metric spaces. The result, concerning set-valued strict contractions (in which the contractive coefficient is not constant), is different from the result of Yuan concerning the existence of endpoints of Tarafdar–Vyborny generalized contractions (in which the contractive coefficient is constant) in bounded metric spaces and provides some examples of Tarafdar–Yuan topological contractions in compact uniform spaces. Definitions and results presented here are new for set-valued dynamical systems in uniform, locally convex and metric spaces and even for single-valued maps. Examples show a fundamental difference between our results and the well-known ones.

MSC: 54C60; 47H10; 54E15; 54E50

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1. Introduction

The Banach contraction principle [5] has assumed a central role in many fields of mathematics and applied mathematics and various concepts of single-valued and set-valued contractions and different results, which essentially generalize the Banach contraction principle, were introduced and established in many papers and books. In particular, Meir and Keeler, using some result of Chu and Diaz [11], established the following result which generalizes the results of Banach [5], Edelstein [12], Rakotch [25] and Boyd and Wong [7] and is very essential in single-valued fixed point theory.

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Theorem 1.1 (*Meir and Keeler [20]*). *Let (X, d) be a complete metric space and let $T : X \to X$ be a map satisfying the condition*

$$\forall_{\varepsilon>0}\exists_{\eta>0}\forall_{x,y\in X}\{[\varepsilon \le d(x, y) < \varepsilon + \eta] \Rightarrow [d(T(x), T(y)) < \varepsilon]\}.$$

Then T has a unique fixed point v in X and the sequence $\{T^{[m]}(w)\}$ converges to v for each $w \in X$.

Maps $T : X \to X$ satisfying this condition are called in the literature *Meir–Keeler contractions* (*MKC*, for short) on X. The above condition and result were sources of further investigations in fixed point theory of single-valued maps. For details, see, e.g., [1,14,15,19,22,23,26,27,31] and references therein.

The asymptotic fixed point theory of single-valued maps in metric spaces involves assumptions about the iterations of a map in question, it uses ideas of Banach, has a long history in nonlinear analysis (see, e.g., [2,8–10,13,16,17, 21,24,28] and references therein) and was initiated by Caccioppoli [9]. In this theory the following definition is well known.

Definition 1.1 (*Kirk [17, Definition 2.1]*). Let (X, d) be a metric space. Then $T : X \to X$ is called an *asymptotic contraction* (*AC*, for short) on X if there exist maps $\varphi_m, \varphi : [0, \infty) \to [0, \infty)$, $m \in \mathbb{N}$, such that:

(i) φ_m are continuous on $[0, \infty)$, $m \in \mathbb{N}$;
(ii) $\varphi(0) = 0$;
(iii) $\forall_{\varepsilon>0}\{\varphi(\varepsilon) < \varepsilon\}$;
(iv) $\{\varphi_m\}$ converges uniformly to φ on $D = \{d(x, y) : x, y \in X\}$; and
(v) $\forall_{x,y\in X}\forall_{m\in\mathbb{N}}\{d(T^{[m]}(x), T^{[m]}(y)) \le \varphi_m(d(x, y))\}$.

The result, concerning asymptotic contractions, given by Kirk, which is an asymptotic version of the Boyd–Wong result [7], reads as follows:

Theorem 1.2 (*Kirk [17, Theorem 2.1]*). *Assume that:*

(i) *(X, d) is a complete metric space;*
(ii) *$T : X \to X$ is an AC;*
(iii) *T is continuous on X; and*
(iv) *there exists $u \in X$ such that the orbit $\{T^{[m]}(u) : m \in \mathbb{N}\}$ of u is bounded.*

Then: (v) *T has a unique fixed point v in X; and* (vi) *the sequence $\{T^{[m]}(w)\}$ converges to v for each $w \in X$.*

In [17], Lim introduced the notion of an L-map and characterized MKC.

Definition 1.2 (*Lim [19]*). A map $\varphi : [0, \infty) \to [0, \infty)$ is called an L-map if:

(i) $\varphi(0) = 0$;
(ii) $\forall_{\varepsilon>0}\{\varphi(\varepsilon) > 0\}$; and
(iii) $\forall_{\varepsilon>0}\exists_{\eta>0}\forall_{t\in[\varepsilon,\varepsilon+\eta]}\{\varphi(t) \le \varepsilon\}$.

Proposition 1.1 (*Lim [19]*). *Let (X, d) be a metric space and let $T : X \to X$. Then T is an MKC if and only if there exists an (nondecreasing, right continuous) L-map φ such that*

$$\forall_{x,y\in X}\{[d(x, y) > 0] \Rightarrow [d(T(x), T(y)) < \varphi(d(x, y))]\}.$$

Very recently, Suzuki [28] introduced a new concept of a single-valued asymptotic contraction of Meir–Keeler type on a metric space and proved, using Definition 1.2 and Proposition 1.1 of Lim [19], a new fixed point theorem for such a contraction.

Definition 1.3 (*Suzuki [28, Definition 4]*). Let (X, d) be a metric space. Then $T : X \to X$ is called an *asymptotic contraction of Meir–Keeler type* (*ACMK*, for short) on X if there exists a sequence $\{\varphi_m\}$ of maps $[0, \infty)$ into itself such that:

(i) $\forall_{\varepsilon\ge0}\{\limsup_m \varphi_m(\varepsilon) \le \varepsilon\}$;