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Entropy solutions for a fourth-order nonlinear degenerate problem for noise removal[☆]

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Abstract

In this paper, we establish the existence and uniqueness of entropy solutions for a fourth-order nonlinear degenerate parabolic problem for noise removal in dimension $1 \leq d < 4$.

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Keywords: Entropy solutions; Fourth-order; Nonlinear degenerate parabolic equation; Noise removal

1. Introduction

In this paper, we consider the following boundary value problem for the fourth-order nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t} + D_{ij}^2 \left(\frac{D_{ij}^2 u}{|D^2 u|} \right) = 0, \quad (x,t) \in Q_T = (0,T) \times \Omega, \tag{1.1}$$

$$u(x,t) = 0, \quad (x,t) \in (0,T) \times \partial\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial N}(x,t) = 0, \quad (x,t) \in (0,T) \times \partial\Omega, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.4}$$

where Ω is a bounded open domain of $\mathbb{R}^d$ with appropriate smooth boundary, $1 \leq d < 4$, $T > 0$ is fixed, $D^2 u \in \mathbb{R}^{d\times d}$ is the Hessian matrix and $|D^2 u| = \sqrt{\sum_{i,j=1}^d |D_{ij}^2 u|^2}$, N denotes the unit outward normal to the boundary $\partial\Omega$. This problem is very useful in image processing for noise removal and preventing staircasing, where $u = u(x,t)$ represents the original image describing a real scene (the unknown), and u_0 is the observed image (the data).

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The use of partial differential equations for image processing has become a major research topic in the past few years. One of the most popular and successful methodologies for image restoration is the use of the total variation minimizing model which was firstly introduced by Rudin et al. [18]. However, because of penalizing little discontinuities corresponding to edges, it possesses the staircasing property which may be undesirable under some circumstances [8,20]. A remarkable approach to overcoming this flaw is to introduce higher order derivatives into the energy [8,17,24]. In [17], one hopes to minimize the energy

$$\text{Minimize} \int_\Omega |D^2 u| \tag{1.5}$$

with some constraints. So constant gradient is allowed, which suppresses staircasing and the energy can be formally identified by the Euler–Lagrange equation (1.1).

Though the effectiveness of fourth-order diffusion equations for noise removal has been shown [8,9,16,17,24], very little is known about the theoretical analysis for the equations. We refer the reader to [15] for traveling wave solutions in one dimension, [16] for weak solution by variational methods and [22] for a generalized thin film equation. Some qualitative properties of solutions for this class of fourth-order diffusion equations (such as extinction in finite time, convergence of the implicit Euler semidiscretization, and so on) can be found in [12].

In this paper, due to the fact that Eq. (1.1) has a strongly degenerate term, we have to introduce a notion of entropy solution. And we present analysis of existence and uniqueness of the entropy solutions. In general we follow the approach considered by Andreu et al., who study minimizing total variation flow $u_t = \mathrm{div}(Du/|Du|)$ with L^1-data [1–6]. They gave the proper definition of the entropy solution and the proof of existence and uniqueness. However we emphasize that the realization of this approach has some distinctive features in our case. They are connected with some particularities of high order equations in comparison with the second-order ones and with the degenerate character of our equations as well. A typical feature of all the works as regards the proof of the existence of a solution for TV flow is to apply the Crandall–Liggett semigroup theory. However, it seems that this is not suitable for high order nonlinear equations. We investigate the existence of the entropy solutions by employing the difference and variation methods inspired by an idea from [23], where an existence result of Young measure solutions of a class of forward–backward diffusion equations is proved. For the functional (1.5) depending on higher derivatives, the natural space for the solutions is the space of functions of the bounded Hessian $BH(\Omega)$, which is developed by F. Demengel in [10,11]. For some references concerning the BH space, we refer the reader to [7,10,11,13,14,19].

It worth mentioning that the boundary conditions for the natural choice are rather complicated and we refer the reader to [17]. For convenience, we choose the zero boundary conditions, which corresponds to padding the boundary of the image with black. However, the proof of this paper is also suitable for natural boundary conditions.

The plan of the paper is the following. In Section 2, we state some preliminaries and the definition of the entropy solutions. Section 3 is devoted to the proofs of our main results.

2. Preliminaries and the statement of the main result

In the following we recall some basic notation and facts concerning the space of $BH(\Omega)$.

Denote by $\mathcal{M}(\Omega, \mathbb{R}^{d\times d})$ the space of the bounded measures on Ω with values in $\mathbb{R}^{d\times d}$ and by $|\cdot|_T$ the total variation of a measure of $\mathcal{M}(\Omega, \mathbb{R}^{d\times d})$, i.e.

$$|\mu|_T = \sup\left\{\int_\Omega \sum_{ij} \varphi_{ij}\mathrm{d}\mu_{ij};\, \varphi_{ij} \in C_0^0(\Omega),\, \sum_{ij}\varphi_{ij}^2 \le 1 \text{ in } \Omega\right\},$$

where $C_0^0(\Omega)$ denotes the closure of continuous functions with compact support in Ω.

We introduce the space of functions with bounded Hessian in Ω

$$BH(\Omega) = \{u \in W^{1,1}(\Omega);\, D^2 u \in \mathcal{M}(\Omega, R^{d\times d})\},$$

where D^2u denotes the distributional Hessian of u. So if $u \in BH(\Omega)$, we have $u \in W^{1,1}(\Omega)$ and

$$|D^2 u|_T = \sup\left\{\int_\Omega \sum_{ij} \varphi_{ij}\cdot u\mathrm{d}x;\, \varphi_{ij} \in C_0^2(\Omega),\, \sum_{ij}\varphi_{ij}^2 \le 1 \text{ in } \Omega\right\}.$$