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Nonlocal second-order geometric equations arising in tomographic reconstruction

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Abstract

In this paper, we study new nonlocal geometric equations which are related to tomographic reconstruction when using the level-set approach. We treat two additional difficulties which make the work original. On one hand, the level lines do not evolve along normal directions and the nonlocal term is not of "convolution type". On the other hand, the speed is not necessarily bounded compared to the nonlocal term. We prove an existence and uniqueness result for our equation.

Keywords: Nonlocal Hamilton–Jacobi equations; Tomographic reconstruction; Nonlocal front propagation; Level-set approach; Viscosity solutions

In this paper, we study a fully nonlinear parabolic equation with nonlocal term, more precisely,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = F(x,t,Du,D^2u,K^{k,+}_{x,t,u}) & \text{in } \mathbb{R}^N \times [0,T], \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1}$$

where $N \geq 1$ is an integer, $T > 0$. The unknown function is $u : \mathbb{R}^N \times [0,T] \longrightarrow \mathbb{R}$, Du and D^2u denote respectively the gradient and the Hessian of u with respect to the space variable, $u_0 \colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is the given initial data, and $K^{k,+}_{x,t,u}$ denotes the nonlocal term given for some integer $k \leq N$ by

$$K^{k,+}_{x,t,u} = [u]^+_{x,t} \cap A^k_x$$

where

$$[u]^+_{x,t} = \left\{ y \in \mathbb{R}^N : u(y,t) \geq u(x,t) \right\}$$

and

$$A^k_x = \left\{ y = (y_1, \ldots, y_k, y_{k+1}, \ldots, y_N) \in \mathbb{R}^N : \pi_k(y) = \pi_k(x) \right\}$$

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where π_k denotes the projection function from $\mathbb{R}^N$ into $\mathbb{R}^k$ defined by

$$\pi_k(x) = (x_1, \dots, x_k)$$

for all $x = (x_1, \dots, x_k, x_{k+1}, \dots, x_N)$. In the same manner

$$K^{k,-}_{x,t,u} = [u]^-_{x,t} \cap A^k_x$$

where

$$[u]^-_{x,t} = \left\{ y \in \mathbb{R}^N : u(y,t) > u(x,t) \right\}.$$

The nonlinearity F is a continuous function from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\} \times \mathscr{S}_N \times \mathscr{B}_{N-k}$ into $\mathbb{R}$, where $\mathscr{S}_N$ is the set of real symmetric $N \times N$ matrices and $\mathscr{B}_{N-k}$ is the set of equivalence classes of all subsets of $\mathbb{R}^{N-k}$ with respect to the relation $A \sim B$ if $\mathcal{L}^{N-k}(A \Delta B) = 0$, where $\mathcal{L}^{N-k}$ is the Lebesgue measure on $\mathbb{R}^{N-k}$.

We consider $\mathscr{B}_{N-k}$ with a topology that comes from the metric

$$d(A, B) = \sum_{n=0}^{\infty} \frac{\mathcal{L}^{N-k}\left((A \Delta B) \cap B(0,n)\right)}{2^n \mathcal{L}^{N-k}(B(0,n))},$$

where $B(0,n)$ denotes the ball in $\mathbb{R}^{N-k}$ of center 0 and radius n. With this topology a sequence $(K^k_n)_{n \geq 1}$ in $\mathscr{B}_{N-k}$ converges to $K^k \in \mathscr{B}_{N-k}$ if and only if $\mathbb{1}_{K^k_n}$ converges to $\mathbb{1}_{K^k}$ in $L^1_{loc}(\mathbb{R}^{N-k})$.

Here, we give in dimension 2 an example of an equation which is related to tomographic reconstruction when using active curves and the level-set approach [12,7] and which has the same type of nonlocal term as Eq. (1). The model case that we have in mind is

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = C(x,t)\left(\displaystyle\int_{K^{1,+}_{x,t,u}} g(z)\mathrm{d}z\right)|Du| + \mathrm{Trace}\left[\left(I - \dfrac{Du \otimes Du}{|Du|^2}\right) D^2 u\right], \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^2 \times \{0\}, \end{cases} \tag{2}$$

where $x = (x_1, x_2)$, g is a positive function and $g \in L^1(\mathbb{R})$, C is a Lipschitz continuous function, and $K^{1,+}_{x,t,u}$ is given by

$$\begin{aligned} K^{1,+}_{x,t,u} &= \left\{ y = (y_1, y_2) \in \mathbb{R}^2 : u(y,t) \geq u(x,t) \right\} \cap \{y_1 = x_1\} \\ &= [u]^+_{x,t} \cap A^1_x. \end{aligned}$$

We recall that the level-set approach was first introduced by Osher and Sethian [13] for numerical computations. It was then developed from a theoretical point of view by Evans and Spruck [10] for motion by mean curvature and by Chen, Giga and Goto [8] for general normal velocities. We also refer the reader to Barles, Soner and Souganidis [3] and Souganidis [15,16] for different presentations and other results on the level-set approach.

In [14], Slepčev studied the motion of fronts in bounded domains with normal velocities which can depend on the nonlocal terms, in addition to the curvature, the normal direction and the location of the front. In fact, the velocities depend on the nonlocal terms if the velocities at any point of the front depend on the set that the front encloses. Depending on the velocities, the motion of the front can be described by the partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = F(x, t, Du(x,t), D^2u(x,t), [u]^+_{x,t}) \quad \text{in } O \times [0,T] \tag{3}$$

with the Neumann boundary conditions $\partial u / \partial \gamma = 0$ on $\partial O \times [0,T]$ and $u(x,0) = u_0(x)$, where O is a bounded domain in $\mathbb{R}^{\mathbb{N}}$. Slepčev proved an existence and uniqueness result for this equation, using the viscosity solution.

In [2], Barles, Cardaliaguet, Ley and Monneau studied the first-order nonlocal equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \left(C_0(.,t) * \mathbb{1}_{[u]^+_{x,t}}(x,t) + C_1(x,t)\right)|Du| \quad \text{in } \mathbb{R}^N \times [0,T], \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N \times \{0\}. \end{cases} \tag{4}$$