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Nonlinear Analysis 70 (2009) 1776–1796

Nonlinear Analysis

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Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton–Jacobi–Bellman equations☆

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Received 25 May 2005; accepted 25 February 2008

Abstract

In this paper we study stochastic optimal control problems with jumps with the help of the theory of Backward Stochastic Differential Equations (BSDEs) with jumps. We generalize the results of Peng [S. Peng, BSDE and stochastic optimizations, in: J. Yan, S. Peng, S. Fang, L. Wu, Topics in Stochastic Analysis, Science Press, Beijing, 1997 (Chapter 2) (in Chinese)] by considering cost functionals defined by controlled BSDEs with jumps. The application of BSDE methods, in particular, the use of the notion of stochastic backward semigroups introduced by Peng in the above-mentioned work allows a straightforward proof of a dynamic programming principle for value functions associated with stochastic optimal control problems with jumps. We prove that the value functions are the viscosity solutions of the associated generalized Hamilton–Jacobi–Bellman equations with integral-differential operators. For this proof, we adapt Peng's BSDE approach, given in the above-mentioned reference, developed in the framework of stochastic control problems driven by Brownian motion to that of stochastic control problems driven by Brownian motion and Poisson random measure.

MSC: 60H10; 60H05; 65C30

Keywords: Poisson random measure; Value function; Backward stochastic differential equations; Dynamic programming principle; Viscosity solution; Stochastic backward semigroup

1. Introduction

Non-linear BSDEs in the framework of Brownian motion were first introduced by Pardoux and Peng [9] in 1990. They have been studied since then by many authors and have found various applications, namely in stochastic

☆ The work has been supported by the NSF of PR China (Nos 10701050; 10671112), Shandong Province (No. Q2007A04), Postdoctoral Science Foundation of Shanghai grant (No. 06R214121) and National Basic Research Program of China (973 Program) (Nos 2007CB814904; 2007CB814906).

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0362-546X/$ - see front matter
doi:10.1016/j.na.2008.02.080

control, finance, and second order partial differential equations (PDE) theory. Non-linear BSDEs in the framework of Brownian motion and Poisson random measure were first introduced by Tang and Li [16] in 1994.

On the other hand, the recursive utility functions defined from BSDEs are important because they can be index scaling risks in the study of economics and finance (see El Karoui, Peng and Quenez [8], Chen and Epstein [4], Duffie and Epstein [5,6], Duffie, Geoffard and Skiadias [7], etc). In [13], Peng gave a probabilistic representation for systems of second order quasilinear parabolic PDEs which generalized the famous Feynman–Kac formula to the non-linear case. There has been much research on relationships between BSDEs and PDEs, for example, by El Karoui, Peng and Quenez [8], Peng [12,14,15], Pardoux and Peng [10], Yong and Zhou [17], etc. Using the notion of viscosity solution, Peng [11,12], studied stochastic optimization theory of classical BSDEs and obtained a dynamic programming principle (DPP). Using this principle, he proved the existence of the viscosity solution of a type of generalized H–J–B equations with only differential operators. Barles, Buckdahn and Pardoux [1] considered BSDEs with jumps and proved the existence and uniqueness of viscosity solutions for a type of integral–differential PDEs, but, with no controls. With controls in the framework of Brownian motion and Poisson random measure it becomes more hard to give the probabilistic representation for systems of generalized fully non-linear second order H–J–B equations with integral–differential operators. In this paper we will investigate them with the help of the theory of BSDEs with jumps and prove the existence of their viscosity solutions. First we investigate the stochastic optimization theory of BSDEs with jumps and define an index functional from a BSDE with jump which is similar to a recursive utility function. Taking the supremum over its controls, we get a value function. Using the dynamic programming principle we obtained, we prove straightforward the existence of a viscosity solution for a type of generalized integral–differential H–J–B equation, i.e., the value function is its viscosity solution.

Our paper is organized as follows. Section 2 and the Appendix recall some elements of the theory of backward SDEs with jumps and forward–backward SDEs with jumps which will be needed in the remaining sections. Section 3 introduces the setting of stochastic optimal control problems with value functions W, and proves that they are deterministic and satisfy the DPP. In Section 4, we make full use of the DPP and Peng's method to prove that W is the viscosity solution of the associated integral–differential H–J–B equation.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space where $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is a filtration satisfying the usual conditions. $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the following two mutually independent stochastic processes:

(i) a d-dimensional standard Brownian motion $\{B_t\}_{t\geq 0}$;

(ii) a Poisson random measure μ on $\mathbb{R}_+ \times E$, where $E = \mathbb{R}^l \setminus \{0\}$ is equipped with its Borel field $\mathcal{B}(E)$. The compensator of μ is $\hat{\mu}(\mathrm{d}t, \mathrm{d}e) = \mathrm{d}t\lambda(\mathrm{d}e)$ which makes $\{\tilde{\mu}((0, t] \times A) = (\mu - \hat{\mu})((0, t] \times A)\}_{t\geq 0}$ a martingale for all $A \in \mathcal{B}(E)$ satisfying $\lambda(A) < \infty$. Here λ is an arbitrarily given σ-finite Lévy measure on $(E, \mathcal{B}(E))$, i.e., a measure on $(E, \mathcal{B}(E))$ with the property that $\int_E (1 \wedge |e|^2)\lambda(\mathrm{d}e) < \infty$. We set T to be an arbitrarily prescribed positive number and we call $[0, T]$ the *time duration*. Obviously, we have

$$\mathcal{F}_t = \sigma\left[\int\int_{(0,s]\times A} N(\mathrm{d}s, \mathrm{d}z) : s \leq t, A \in \mathcal{B}(E)\right] \vee \sigma[W_s : s \leq t] \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of P-null sets.

For any $n \geq 1$, $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$. We also shall introduce the following three spaces of processes which will be used frequently throughout the paper:

$$\mathcal{S}^2(0, T; \mathbb{R}) := \{(\psi_t)_{0\leq t\leq T} \text{ real-valued } \mathbb{F}\text{-adapted càdlàg process} : E[\sup_{0\leq t\leq T} |\psi_t|^2] < +\infty\};$$

$$\mathcal{H}^2(0, T; \mathbb{R}^n) := \left\{(\psi_t)_{0\leq t\leq T} \mathbb{R}^n\text{-valued } \mathbb{F}\text{-progressively measurable process} : \|\psi\|^2 = E\left[\int_0^{\mathrm{T}} |\psi_t|^2 \mathrm{d}t\right] < +\infty\right\};$$