Nonlinear Analysis 71 (2009) 1485–1495

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Universality with respect to ω-limit sets

Jacek Chudziak [a], Juan Luis García Guirao [b], L'ubomír Snoha [c], Vladimír Špitalský [c,*]

[a] Department of Mathematics, University of Rzeszów, Rejtana 16 A, 35-310 Rzeszów, Poland

[b] Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, 30203 Cartagena (Murcia), Spain

[c] Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia

ARTICLE INFO

Article history:
Received 19 March 2008
Accepted 8 December 2008

MSC:
primary 37B20
secondary 37B45
37E25
54H20

Keywords:
ω-limit set
Universal system
Graph
Dendrite
Cantor set

ABSTRACT

A discrete dynamical system on a compact metric space X is called universal (with respect to ω-limit sets) if, among its ω-limit sets, there is a homeomorphic copy of any ω-limit set of any dynamical system on X. By a result of Pokluda and Smítal the unit interval admits a universal system. In this paper, we study the problem of the existence of universal systems on Cantor spaces, graphs, dendrites and higher-dimensional spaces.

1. Introduction

In the present paper a *dynamical system* is a pair (X, f), where X is a compact metric space and f is a continuous selfmap of X. Given a point $x \in X$, its *trajectory* is the sequence $(f^n(x))_{n=0}^{\infty}$ and its *orbit* is the set $\{f^n(x) : n = 0, 1, 2, \dots\}$, where $f^n = f^{n-1} \circ f$ if $n \geq 1$ and f^0 is the identity on X.

When studying a dynamical system, one of the main aims is to describe the asymptotic behavior of its trajectories. To do this we use the notion of the ω-limit set. The *ω-limit set of f at a point* $x \in X$ (or the ω-limit set of x under f) is the set $\omega_f(x)$ of all limit points of the trajectory $(f^n(x))_{n=0}^{\infty}$. This set is nonempty, closed and strongly invariant (i.e., $f(\omega_f(x)) = \omega_f(x)$). Since X is compact the ω-limit set $\omega_f(x)$ is the smallest closed set, such that each of its neighborhoods contain all but finitely many points from the trajectory $(f^n(x))_{n=0}^{\infty}$.

The topological characterization of ω-limit sets is known only in some simple spaces (we refer the reader to Section 2.1 for definitions). For the interval, it was found in [1] (one can see also [2]), on the circle in [3] and on all graphs in [4]. On a graph, the ω-limit sets are nonempty nowhere dense closed subsets and finite unions of nondegenerate closed subgraphs. This means that any ω-limit set of any dynamical system on a given graph G is of one of these two types and, conversely, for any such subset M of G there exists a continuous map $f : G \to G$ and a point $x \in G$ with $\omega_f(x) = M$. On hereditarily locally connected continua, the characterization of ω-limit sets is more complicated, see [5] (the ω-limit sets of *homeomorphisms* on dendrites were characterized in [6]). Apart from these results, the topological characterization of ω-limit sets is known

* Corresponding author.
E-mail addresses: chudziak@univ.rzeszow.pl (J. Chudziak), juan.garcia@upct.es (J.L. García Guirao), snoha@fpv.umb.sk (L'. Snoha), spitalsk@fpv.umb.sk (V. Spitalský).

0362-546X/$ – see front matter
doi:10.1016/j.na.2008.12.034

only in zero-dimensional spaces (we recall this apparently folklore result in Theorem 13). In higher-dimensional spaces only partial results are known; we will use results from [7] (for a list of other references see e.g. [5]).

Since the family ω_f of all ω-limit sets of a dynamical system (X, f) is important for the description of the asymptotic behavior of its trajectories, this family is a natural object to study. By [8], for any dynamical system on the interval I, ω_f is a compact set in the space of all nonempty closed subsets of I equipped with the Hausdorff metric. The same is true on the circle and on graphs [3,9], but not on dendrites [10] and on higher-dimensional spaces [11].

The family of all ω-limit sets of all dynamical systems on X is denoted by ω_X. Obviously, ω_X always contains at least all singletons $\{x\}$, $x \in X$. It is possible that no other sets are in ω_X. The examples of such spaces X are, apart from a singleton space, the rigid continua (a compact connected metric space X is called a *rigid continuum* if the only continuous selfmaps of X are the constant maps and the identity). Given a continuous map $\varphi : X \to X$, a natural question is how rich the family ω_φ (of all ω-limit sets of φ) can be compared with the whole family ω_X. In rare cases it is possible that $\omega_\varphi = \omega_X$, i.e., for every continuous map $f : X \to X$ and every $x \in X$ there is $y \in X$ such that $\omega_\varphi(y) = \omega_f(x)$. Then we say that φ is *super universal*. It is easy to verify that X admits a super universal map if and only if the only possible ω-limit sets on X are the singletons—in such a case the identity on X is (the unique) super universal map. Thus the identity on a rigid continuum is a super universal map.

So, the notion of super universality is too restrictive. Bruckner [12, p. 201] asked whether there is a dynamical system (I, φ) on the interval I with the property that for any $M \in \omega_I$ there is a *homeomorphic copy* of M in ω_φ. Some partial results in this direction have been obtained in [13,14]. The question in its full generality was answered affirmatively in [15]. It is remarkable that the map $\varphi : I \to I$ constructed in [15] has the property that for every $M \in \omega_I$ there is a homeomorphism h of the *whole interval* I onto itself such that $h(M) \in \omega_\varphi$.

To shorten formulations we introduce the following terminology. Given a compact metric space X and subsets A and B of X, we say that B is a *homeomorphic copy* of A if there is a homeomorphism $h : A \to B$. In the case when at least one homeomorphism $h : A \to B$ can be extended to a homeomorphism of the whole space X onto itself we say that B is a *strongly homeomorphic copy* of A. (Note that this defines an equivalence relation on subsets of the space.) A dynamical system (X, φ) is called *(strongly) universal* if among its ω-limit sets there is a (strongly) homeomorphic copy of any $M \in \omega_X$. In such a case we will often say that the map φ itself is (strongly) universal. Thus the above mentioned result from [15] can be reformulated by saying that the interval admits a strongly universal map.

Obviously, for a map φ the following implications hold:

$$\text{super universal} \Longrightarrow \text{strongly universal} \Longrightarrow \text{universal}.$$

The converse implications do not hold. There is a strongly universal map which is not super universal [15] and there is a universal map which is not strongly universal (see e.g. Theorem 4).

Being inspired by [15], we are interested in the existence of (strongly) universal maps in other spaces important in dynamics. We concentrate on Cantor spaces, graphs, dendrites and higher-dimensional spaces.

It turns out that if a subspace of Euclidean n-space ($n \geq 2$) admits a strongly universal map then this subspace has to be at most one-dimensional or, if not, it has to look "strange". By this we mean that at no point can it look like Euclidean m-space, $m = 2, 3, \ldots, n$. In fact, our first result says a bit more:

Theorem 1. *No compact metric space X containing an open set homeomorphic to Euclidean m-space ($m \geq 2$) admits a strongly universal map. Moreover, if X is an m-dimensional compact manifold (with or without boundary) then it does not admit a universal map.*

The second part of the theorem can be slightly generalized. Instead of the assumption that X is an m-dimensional compact manifold, it is sufficient to assume that X satisfies the condition considered in Corollary 8 below.

Among one-dimensional spaces we have a definitive result for graphs:

Theorem 2. *Let X be a graph. Then the following are equivalent:*

(1) *X admits a universal map,*
(2) *X admits a strongly universal map,*
(3) *X is an arc.*

The authors do not know whether there is a dendrite different from an arc which admits a universal map. Nevertheless, the following holds:

Theorem 3. *The only nondegenerate dendrite admitting a strongly universal map is the arc.*

Compared to the negative results above, we have the following result for a *Cantor space* (i.e., a totally disconnected compact metric space with no isolated point):

Theorem 4. *A Cantor space admits a universal map but does not admit a strongly universal one.*

The paper is organized as follows. In Section 2 we introduce the notation, recall needed definitions and present some basic properties of ω-limit sets. In Section 3 we prove Theorems 1–3. Finally, in Section 4 we prove Theorem 4.