Nonlinear Analysis 71 (2009) 1562–1571

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Fixed point properties and proximinality in Banach spaces

T. Domínguez Benavides [a], J. García Falset [b], E. Llorens-Fuster [b], P. Lorenzo Ramírez [a,*]

[a] Departamento de Análisis Matemático, Universidad de Sevilla, P.O. Box 1160, 41080-Sevilla, Spain

[b] Departament d'Anàlisi Matemàtica, Universitat de València, Dr. Moliner 50, 46100, Burjassot, València, Spain

ARTICLE INFO

Article history:
Received 1 October 2008
Accepted 14 December 2008

MSC:
47H10
47H09
46B20

Keywords:
Fixed points
Nonexpansive mappings
Mappings admitting a center
Weak star topology

ABSTRACT

In this paper we prove the existence of a fixed point for several classes of mappings (mappings admitting a center, nonexpansive mappings, asymptotically nonexpansive mappings) defined on the closed convex subsets of a Banach space satisfying some proximinality conditions. In particular, we derive a sufficient condition, more general than weak star compactness, such that if C is a bounded closed convex subset of ℓ_1 satisfying this condition, then every nonexpansive mapping $T : C \to C$ has a fixed point.

1. Introduction

Let X be a Banach space and C be a nonempty subset of X. It is commonly said that C has the fixed point property with respect to a class of mappings if every mapping T in this class and defined from C to C has a fixed point. For example the well-known Schauder Theorem states that every compact convex C has the fixed point property for continuous mappings.

In Metric Fixed Point Theory much effort has been spent in identifying classes of sets (closed convex bounded sets, convex weakly compact sets, convex weakly star compact sets, etc.) of some Banach spaces (usually enjoying a nice geometric property) satisfying the fixed point property for nonexpansive mappings.

For example in Hilbert spaces it is known [1] that a closed convex subset has the fixed point property for nonexpansive mappings if and only if it is bounded. Moreover, all contractible finite unions of closed bounded and convex sets, and all closed bounded starshaped sets have this property. Even not necessarily convex sets but with the property of being Chebyshev with respect to its convex closure have the fixed point property for nonexpansive mappings. (See [2] and the references therein).

For non-Hilbertian Banach spaces there are different situations. For instance in the classical space c_0, Dowling, Lennard and Turett [3] have proven that a nonempty, closed, bounded, convex subset of c_0 has the fixed point property for nonexpansive mappings if and only if it is weakly compact, giving an affirmative answer to a conjecture appeared several years before in [4].

However, this result is not true for the Banach space $(\ell_1, \|\cdot\|_1)$. It is well known that the weakly star compact convex subsets of $(\ell_1, \|\cdot\|_1)$ have the fixed point property for nonexpansive mappings (see [5]), but there exist some closed convex and bounded subsets without this property (see [6]).

* Corresponding author. Tel.: +34 954557713; fax: +34 954557972.
E-mail addresses: tomasd@us.es (T. Domínguez Benavides), garciaf@uv.es (J. García Falset), enrique.llorens@uv.es (E. Llorens-Fuster), ploren@us.es (P. Lorenzo Ramírez).

0362-546X/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2008.12.048

Thus it is quite natural to ask: What kind of closed bounded and convex subsets of $(\ell_1, \|\cdot\|_1)$ have the fixed point property for nonexpansive mappings? (Very recently in [7], it has been shown that all closed convex bounded subsets of ℓ_1 satisfy the fixed point property with respect to the class of mappings which are $\|\cdot\|$-nonexpansive, where $\|\cdot\|$ is a norm on ℓ_1 equivalent to $\|\cdot\|_1$). We will give a property of closed convex bounded subsets of ℓ_1, more general than weak star compactness which implies the fixed point property for nonexpansive and asymptotically nonexpansive mappings.

Since we only need to use some properties of ℓ_1 with respect to the weak star topology, which are shared by many other Banach spaces with respect to some other suitable topologies, we develop our results in a more general setting. We consider that τ is a linear topology on a Banach space X, weaker than the norm topology. In our main results, we assume that the Banach space X satisfies a property with respect to τ, called $L(\tau)$, which can be seen as an abstract version of some properties which are satisfied by the spaces ℓ_p, $1 < p < \infty$ with respect to the weak topology or ℓ_1 with respect to the weak star topology. This condition is shared by some function spaces for suitable topologies, for instance, $L_p(\Omega)$ with respect to the topology of local convergence in measure or any L-embedded Banach space with respect to the abstract measure topology.

In Section 2, we state the notation and show some preliminary results which will be needed in the paper. In particular, for a subset C of X we introduce some compactness conditions concerning proximinal subsets of C, which will be called properties (P) and (P_τ).

The class of mappings admitting a center (i.e. $\|Tx - x_0\| \leq \|x - x_0\|$ for some $x_0 \in X$), will be a basic tool in this paper. Some fixed point results for mappings admitting a center were given in [8]. In Section 3 we give a complete characterization of the fixed point existence for mappings admitting a center by means of Property (P).

Section 4 is devoted to study the existence of fixed points in an important class of spaces which are usually called strictly $L(\tau)$-spaces and which contains all Lebesgue $L^p(\Omega)$ spaces for $p \geq 1$. We will show that for these spaces there is a strong connection between mappings admitting a center and nonexpansive mappings, and so we can use the results in Section 3 to obtain new fixed point results for nonexpansive mappings. In particular, we obtain a class of subsets of ℓ_1, more general than the class of the weak star compact convex subsets, which have the fixed point property for nonexpansive mappings. The $L(\tau)$-property also proves to be useful to obtain fixed point results for multivalued nonexpansive mappings and asymptotically nonexpansive mappings and to study behavior of the iterates of a nonexpansive mapping.

Finally, in Section 5, we use the notion of asymptotically isometric basis to give some additional theorems in ℓ_1, improving some results of [9].

2. Preliminaries

Along this paper $(X, \|\cdot\|)$ will be a Banach space and τ a linear topology on X weaker than the norm topology. For a bounded sequence (x_n) in X and a closed convex bounded subset C of X, we denote:

- $\phi_{(x_n)} : X \to \mathbb{R}^+$ the real function $\phi_{(x_n)}(y) := \limsup_{n\to\infty} \|x_n - y\|$.
- Given $x \in X$, $d(x, C) := \inf\{\|x - y\| : y \in C\}$.
- Given $x_0 \in X$, $P_C(x_0) := \{x \in C \ : \ \|x - x_0\| = d(x_0, C)\}$.
- Given $x_0 \in X$, $r > 0$, $B[x_0, r] := \{x \in X : \ \|x_0 - x\| \leq r\}$.

The following concept was introduced in [8]:

Definition 2.1. Let C be a bounded closed convex subset of $(X, \|\cdot\|)$. A point $x_0 \in X$ is said to be a center for a mapping $T : C \to X$ if, for each $x \in C$, $\|Tx - x_0\| \leq \|x - x_0\|$.

Definition 2.2. Let C, A be nonempty subsets of $(X, \|\cdot\|)$. We say that C has the fixed point property for continuous mappings of C with center in A if every continuous mapping $T : C \to C$ admitting a center $x_0 \in A$ has a fixed point.

Definition 2.3 (*See [10]*). We say that X has the τ fixed point property (τ-FPP) if every nonexpansive mapping T defined from a convex norm-bounded τ-sequentially compact subset C of X into C has a fixed point.

Notice that Definition 2.3 is equivalent to the usual definition of w-FPP when τ is the weak topology. When X is the dual of a separable Banach space, it is also equivalent to the w^*-FPP if τ is the weak star topology.

From now on, we refer to $\overline{C}^\tau$ as the sequential closure of C with respect to the topology τ.

Definition 2.4. Let C be a nonempty closed convex bounded subset of $(X, \|\cdot\|)$. Let A be a nonempty subset of X. We say that C has Property (P) with respect to A if for every $x \in A$ the set $P_C(x)$ is a nonempty and norm-compact subset of C. In particular, for simplicity, we will say that C has Property (P) if it has Property (P) with respect to $\overline{C}^\tau$.

Definition 2.5. Let C be a nonempty closed convex bounded subset of $(X, \|\cdot\|)$. We say that C has Property (P_τ) if for every $x \in \overline{C}^\tau$ the set $P_C(x)$ is a nonempty and τ-sequentially compact subset of C.

Obviously, Property (P) implies (P_τ).