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Uniform blow-up profiles for nonlinear and nonlocal reaction–diffusion equations☆

Qilin Liu [a,*], Yichao Chen [b], Shengqi Lu [b]

[a] Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, PR China

[b] Department of Mathematics, Southeast University, Nanjing, 210018, Jiangsu, PR China

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ABSTRACT

In this paper, we investigate the blow-up properties of positive solutions to the following integro-parabolic equations

$$u_t = \Delta u + e^{\alpha u} + \frac{\gamma - 1}{|\Omega|}\int_\Omega e^{\beta u} dx,$$

where $\alpha, \beta > 0$ and $\gamma > 1$, and $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$. For the radially symmetric and non-increasing initial data, we give a complete classification in terms of global and single point blow-up according to the parameters α and β. Moreover, the blow-up rates are also determined in each case. Particularly, for the special case: $\alpha = \beta$ and $n \leq 2$, it will be proved that the blow-up rate at $x = 0$ is faster than that at the other point $x \neq 0$. This seems to be a new phenomenon for this kind of problem.

1. Introduction and main results

The thermal combustion process in a solid fuel can be modelled by the semi-linear parabolic equation [1]

$$u_t = \Delta u + f(u), \tag{1.1}$$

where $f(u) = e^u$ or u^p with $p > 1$. However, for an ideal gaseous fuel in a bounded container, the motion caused by the compressibility of the gas leads to the addition of a nonlocal integral term that complicates the model. For example, the ignition period of a thermal event can be described by the following integro-parabolic problem [2,3,1]

$$\begin{cases} u_t = \Delta u + e^{\alpha u} + \dfrac{\gamma - 1}{|\Omega|}\displaystyle\int_\Omega e^{\beta u} dx, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where α and β are positive constants, $\Omega \subset \mathbb{R}^n$ is a bounded domain (container) with smooth boundary, u is the temperature perturbation of the gas, and $\gamma > 1$ is the gas parameter. When $\gamma = 1$, the problem (1.2) becomes (1.1) that has received much attention, see [1,4,5] and the references therein.

In case that $\alpha = \beta = 1$, Bricher [2] proved that the solution blows up everywhere if $n \leq 2$, and blows up at a single point if $n \geq 3$ and $\gamma - 1 > 0$ is sufficiently small. However, the blow-up set is not clear for $n \geq 3$ and the general $\gamma > 1$. When

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* Corresponding author.
E-mail addresses: liuqlseu@yahoo.com.cn (Q. Liu), chenyc5985@sohu.com (Y. Chen), Lusq@sju.js.cn (S. Lu).

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u blows up everywhere we also say that u is a global blow-up solution, and when u blows up at a single point we also say that u is a single point blow-up solution. A number of papers have considered similar reaction–diffusion equations with a competition between local and nonlocal terms and studied its effect on blow-up sets and rates. Let us mention, for instance References [6–12].

In this paper, we study the blow-up properties of the solution to the problem (1.2) for the radial case. Let $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$. We first state two assumptions.

(A1) $u_0 \in C^2(\Omega),\ u_0 = u_0(r)$ with $r = |x|,\ u_0'(r) < 0$ for $0 < r \le R,\ u_0(R) = 0$ and $u_0''(0) < 0$.
(A2) $\Delta u_0 + e^{\alpha u_0} + \frac{\gamma-1}{|\Omega|}\int_\Omega e^{\beta u_0}dx \ge 0, \quad x \in \Omega$.

Remark 1.1. (i) If u_0 satisfies the assumption (A1), then $u_0(r) > 0$ for $0 \le r < R$.

(ii) Assume that $\varphi_0(x)$ satisfies the condition (A1). Then the function $u_0(x) = \lambda\varphi_0(x)$ satisfies the condition (A2) provided that $\lambda > 0$ is large enough.

(iii) If the initial data u_0 satisfies the conditions (A1) and (A2), then the solution $u(x, t) = u(r, t)$ is radially symmetric and satisfies $u_r \le 0,\ u_t \ge 0$.

Let u be the blow-up solution of (1.2) and set

$$g(t) = \frac{\gamma - 1}{|\Omega|}\int_\Omega e^{\beta u}dx, \qquad G(t) = \int_0^t g(s)ds.$$

Our main results are the following:

Theorem 1.1. *Let $\varphi_0(x)$ be a fixed function and satisfy the condition* (A1). *Set $u_0(x) = \lambda\varphi_0(x)$. If $\beta > \alpha$, then there exists a constant $\Lambda(\varphi_0) > 0$, such that*

$$\lim_{t\to T^*}\frac{u(x,t)}{|\ln(T^*-t)|} = \frac{1}{\beta}, \quad \forall x \in \Omega$$

provided that $\lambda > \Lambda(\varphi_0)$. This implies that u blows up uniformly on compact subset of Ω.

Theorem 1.2. *Assume that the conditions* (A1) *and* (A2) *hold. If $\alpha = \beta$, then we have*

(i) *if $n \le 2$, then the solution $u(x, t)$ blows up everywhere and has the following non-uniform blow-up rates:*

$$\lim_{t\to T^*}\frac{u(0,t)}{|\ln(T^*-t)|} = \frac{1}{\alpha},$$
$$\lim_{t\to T^*}\frac{u(x,t)}{|\ln(T^*-t)|} = 0, \quad \forall x \in \Omega \setminus \{0\};$$

(ii) *if $n \ge 3$, then the solution $u(x, t)$ blows up only at the single point $x = 0$ and satisfies*

$$\lim_{t\to T^*}\frac{u(0,t)}{|\ln(T^*-t)|} = \frac{1}{\alpha}.$$

Theorem 1.3. *Assume that the assumptions* (A1) *and* (A2) *hold. If $\alpha > \beta$, then the solution $u(x, t)$ blows up only at the single point $x = 0$ and satisfies*

$$\lim_{t\to T^*}\frac{u(0,t)}{|\ln(T^*-t)|} = \frac{1}{\alpha}.$$

Remark 1.2. The result (i) of Theorem 1.2 shows that, when $\alpha = \beta$ and $n \le 2$, the solution $u(x, t)$ blows up everywhere and the blow-up rate at $x = 0$ is faster than that at the other point $x \ne 0$. It seems to be a new and surprising phenomenon for this kind of problem. However, it is not clear if the blow-up rate is still uniform outside of the origin. It is an interesting open problem to determine this rate.

We organize our paper as follows. Section 2 is devoted to the proof of Theorem 1.1. In Section 3, we prove Theorems 1.2 and 1.3.

2. The proof of Theorem 1.1

We first give the following lemmas which will be used in the sequel.

Lemma 2.1 (*[3]*). *Assume* (A1) *holds.*

(i) *If $\lim_{t\to T^*} G(t) = \infty$, then blow-up occurs everywhere.*
(ii) *If $\lim_{t\to T^*} G(t) < \infty$, then blow-up occurs only at the origin.*
As a consequence, the blow-up solution of (1.2) blows up either everywhere or at the single point $x = 0$.