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Reduction principle and dynamic behaviors for a class of partial functional differential equations☆

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ABSTRACT

We investigate the dynamic of solutions in the α-norm for some nonhomogeneous linear partial functional differential equations. We suppose that the undelayed homogeneous part is the infinitesimal generator of an analytic semigroup, and the delayed part is continuous with respect to fractional powers of the generator. We establish a reduction principle for the infinite dimensional system in order to reduce its qualitative analysis to a finite dimensional one. Our reduction method is based on a new variation of constants formula. In application, the reduced system is used to prove the existence of almost automorphic, almost periodic and periodic solutions for the whole infinite dimensional system.

1. Introduction

The reduction principle for infinite dynamical systems is an interesting tool to reduce their complexity. Frequently, a model formulated in terms of partial differential equations (PDE) can be reduced to an ordinary differential equation (ODE) in finite dimensional space. This ODE is usually obtained by projecting the PDE on a finite dimensional subspace which retains most relevant features of the whole system. At any rate the asymptotic behavior of the original system is the same as that of the reduced one.

In this work, we study the behavior of the following class of partial functional differential equations

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t) & \text{for } t \geq \sigma, \\ u_\sigma = \varphi \in \mathcal{C}_\alpha := \mathcal{C}([-r, 0]; X_\alpha), \end{cases} \tag{1.1}$$

where the operator $-A : D(A) \to X$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $(X, |.|)$. For $0 < \alpha \leq 1$, X_α denotes the Banach space $D(A^\alpha)$ endowed with the norm

$$|x|_\alpha = |A^\alpha x| \quad \text{for } x \in D(A^\alpha),$$

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where A^α is the fractional power of A which will be defined below. $\mathcal{C}_\alpha := \mathcal{C}([-r, 0]; X_\alpha)$ is the space of continuous functions from $[-r, 0]$ to X_α endowed with the uniform norm topology

$$|\phi|_{\mathcal{C}_\alpha} = \sup_{-r\le\theta\le 0} |\phi(\theta)|_\alpha.$$

As usual, for every $\sigma \in \mathbb{R}$ and $t \ge \sigma$, the history function $u_t \in \mathcal{C}_\alpha$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in [-r, 0].$$

L is a bounded linear operator from $\mathcal{C}_\alpha$ into X and f is a continuous function from $[\sigma, +\infty)$ into X. As a model for the class (1.1) one may take the following system which describes many phenomena in physical systems

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) = \displaystyle\sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial}{\partial x_j} v(t,x)\right) - a_0 v(t,x) + \varepsilon \sum_{i=1}^{n} \dfrac{\partial}{\partial x_i} v(t-r_i, x) \\ \qquad\qquad + \int_{-r_{n+1}}^{0} \beta(\theta) v(t+\theta, x)\mathrm{d}\theta + \Theta(t,x) \quad \text{for } t \ge 0 \quad \text{and} \quad x \in \Omega, \\ v(t,x) = 0 \quad \text{for } t \ge 0 \quad \text{and} \quad x \in \partial\Omega, \\ v(\theta, x) = \varphi_0(\theta, x) \quad \text{for } \theta \in [-r, 0] \quad \text{and} \quad x \in \Omega, \end{cases} \tag{1.2}$$

where a_0, ε are constants, $r = \max\{r_1, \ldots, r_{n+1}\}$, Ω is an open bounded set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\beta \in L^2([-r_{n+1}, 0], \mathbb{R})$, $\Theta : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is continuous, the initial function $\varphi_0 : [-r, 0] \times \Omega \to \mathbb{R}$ is a given function.

The theory and applications of partial functional differential equations are an active research area. It has been extensively studied in the past years (see [1–7] and the references therein). Additionally, such equations can exhibit a very rich behavior like almost periodicity, almost automorphy or periodicity (see [8,9]). Since 1974, Travis and Webb [10–12] established the basic theory for the existence and stability of solutions for the following equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t) = -Au(t) + G(t, u_t) \quad \text{for } t \ge \sigma, \\ u_\sigma = \varphi \in \mathcal{C}_\alpha := \mathcal{C}([-r, 0]; X_\alpha), \end{cases}$$

where $G : [\sigma, +\infty) \times X \to X$ is a nonlinear continuous function, as application the authors proposed in [11] the following model

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) = \dfrac{\partial^2}{\partial x^2} v(t,x) + F\left(v(t-r, x), \dfrac{\partial}{\partial x} v(t-r, x)\right) \quad \text{for } t \ge 0 \quad \text{and} \quad x \in [0, \pi], \\ v(t, 0) = v(t, \pi) = 0 \quad \text{for } t \ge 0, \\ v(\theta, x) = \varphi_0(\theta, x) \text{for } \theta \in [-r, 0] \quad \text{and} \quad x \in [0, \pi], \end{cases}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous function.

In this paper, we develop a new reduction principle for Eq. (1.1) to prove that the dynamic of bounded solutions on $\mathbb{R}$ is governed by an ordinary differential equation in a finite dimensional space. This goal will be done through a new variation of constants formula. As application, we propose to study the existence of almost automorphic, almost periodic and periodic solutions of Eq. (1.1). In the periodic case, usually we use the fixed point theory to prove that the Poincaré map associated with the equation has a fixed point. In the almost automorphic and the almost periodic cases the situation is different and more complicated since the fixed point approach cannot be applied.

Almost automorphic, almost periodic and periodic solutions are interesting phenomena in dynamical systems. Recall that the concept of almost automorphy is more general than the one of almost periodicity. It was introduced by Bochner and studied by many authors. For more details on almost automorphic functions we refer to [13]. Let us recall some well established results in this field. Consider the following ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = Bx(t) + b(t) \quad \text{for } t \in \mathbb{R}, \tag{1.3}$$

where B is a constant $n \times n$-matrix and $b : \mathbb{R} \to \mathbb{R}^n$ is continuous and τ-periodic. In [14], Massera proved the existence of an τ-periodic solution under the existence of a bounded solution on $\mathbb{R}^+$. Bohr and Neugebauer extended Massera's Theorem to almost periodic case (see [15]). Recently, the results in [16] extended Bohr and Neugebauer's Theorem to almost automorphic case for Eq. (1.3). For partial functional differential equations, we refer to [1,3,4,8,9,17,18].

For Eq. (1.1), we use the reduction principle to show the existence of almost automorphic, almost periodic and periodic solutions. Massera's Theorem, Bohr and Neugebauer's Theorem and the results of [16] are extended to Eq. (1.1): the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of an almost automorphic (resp. almost periodic, resp. periodic) solution if the input function f is almost automorphic (resp. almost periodic, resp. periodic).

The organization of this work is as follows: in Section 2, we recall some preliminary results on the fractional powers of unbounded linear operators generating analytic semigroups and some results regarding the existence of solutions for Eq. (1.1). In Section 3, we establish a variation of constants formula for Eq. (1.1). In Section 4, we develop a reduction principle for Eq. (1.1). In Section 5, we prove the existence of almost automorphic, almost periodic and periodic solutions of Eq. (1.1). In the hyperbolic case, we study the uniqueness of almost automorphic, almost periodic and periodic solutions of Eq. (1.1). In the last section, we apply our theoretical results to the partial functional differential equation (1.2).