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Convergence of composite iterative methods for finding zeros of accretive operators☆

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ABSTRACT

Strong convergence theorems on composite iterative schemes by the viscosity approximation methods for finding a zero of an accretive operator are established in Banach spaces. The main results generalize the recent corresponding results of Aoyama et al. [K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in Banach spaces, Nonlinear Anal. 67 (2007) 2350–2360], Ceng et al. [L.C. Ceng, A.R. Khan, Q.H. Ansari, J.C. Yao, Strong convergence of composite iterative schemes for zeros of m-accretive operators in Banach spaces, Nonlinear Anal. 70 (2009) 1830–1840], Kim and Xu [T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations, Nonlinear Anal. 61 (2005) 51–60], and Xu [H.K. Xu, Strong convergence of an iterative method for nonexpansive and accretive operators, J. Math. Anal. Appl. 314 (2006) 631–643] to viscosity methods in a strictly convex and reflexive Banach space having a uniformly Gâteaux differentiable norm. Our results also improve the corresponding results of [T.D. Benavides, G.L. Acedo, H.K. Xu, Iterative solutions for zeros of accretive operators, Math. Nachr. 248–249 (2003) 62–71; R. Chen, Z. Zhu, Viscosity approximation fixed points for nonexpansive and m-accretive operators, Fixed Point Theory Appl. 2006 (2006) 1–10; S. Kamimura, W. Takahashi, Approximation solutions of maximal monotone operators in Hilbert spaces, J. Approx. Theory 106 (2000) 226–240; P.E. Maingé, Viscosity methods for zeroes of accretive operators, J. Approx. Theory 140 (2) (2006) 127–140; K. Nakajo, Strong convergence to zeros of accretive operators in Banach spaces, J. Nonlinear Convex Anal. 7 (2006) 71–81].

1. Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|,\ x,\ y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. That is, $\Sigma_C = \{f : C \to C \mid f \text{ is a contraction with constant } k\}$. Note that each $f \in \Sigma_C$ has a unique fixed point in C. Let $T : C \to C$ be a nonexpansive mapping (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|,\ \ x,\ y \in C$), and $F(T)$ denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$.

Recall that a (possibly multivalued) operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ in E is *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1,\ 2)$, there exists a $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. (Here J is the duality mapping.) An accretive operator A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$. An accretive operator A is

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m-accretive if $R(I + rA) = E$ for each $r > 0$. If A is an accretive operator which satisfies the *range condition*, then we can define, for each $r > 0$ a mapping $J_r : R(I + rA) \to D(A)$ defined by $J_r = (I + rA)^{-1}$, which is called the resolvent of A. We know that J_r is nonexpansive and $F(J_r) = A^{-1}0$ for all $r > 0$. The set of zeros of A is denoted by F. Hence

$$F := \{z \in D(A) : 0 \in Az\} = A^{-1}0.$$

If $A^{-1}0 \neq \emptyset$, then the inclusion $0 \in Az$ is solvable.

We consider an iterative scheme: for resolvent J_{r_n} of m-accretive operator A,

$$x_{n+1} = J_{r_n} x_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in E$ is chosen arbitrarily. The iterative scheme (1.1) has extensively been studied over the last forty years for construction of zeros of accretive operators (see, e.g., [1–7]).

Recently, Kim and Xu [8] and Xu [9] provided a simpler modification of Mann iterative scheme in either a uniformly smooth Banach space or a reflexive Banach space having a weakly sequentially continuous duality mapping for finding a zero of an m-accretive operator A as follows:

$$\begin{cases} x_0 = x \in E, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \end{cases} \tag{1.2}$$

where $u \in \overline{D(A)}$ is an arbitrary (but fixed) element and the sequence $\{\alpha_n\}$ in $(0, 1)$ (see also [10,11]). They proved that $\{x_n\}$ generated by (1.2) converges to a zero of m-accretive operator A under the control conditions:

(H1) $\lim_{n\to\infty} \alpha_n = 0$,
(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$, or, equivalently, $\prod_{n=0}^{\infty}(1 - \alpha_n) = 0$,
(H3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(H4) $r_n \geq \varepsilon$, $(n \geq 0)$, for some $\varepsilon > 0$ and $\sum_{n=1}^{\infty} |1 - \frac{r_{n-1}}{r_n}| < \infty$, or
(H5) $r_n \geq \varepsilon$, $(n \geq 0)$, for some $\varepsilon > 0$ and $\sum_{n=0}^{\infty} |r_n - r_{n-1}| < \infty$.

Very recently, Aoyama et al. [12] studied the following iterative scheme in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm: for resolvents J_{r_n} of an accretive operator A such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ and $\{\alpha_n\} \subset (0, 1)$,

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n. \end{cases} \tag{1.3}$$

They proved that $\{x_n\}$ generated by (1.3) converges strongly to a zero of A under the conditions (H1), (H2) and (H3) and the condition (H5) on $\{r_n\}$. In the case that C is a compact convex subset of a Banach space having a uniformly Gâteaux differentiable norm, Miyake and Takahashi [13] also proved the convergence of $\{x_n\}$ generated by (1.3) to a zero of an accretive operator A such that $D(A) \subset C \subset \bigcap_{r>0} R(I + rA)$ under conditions (H1) and (H2) and $\lim_{n\to\infty} r_n = \infty$.

Recently, Qin and Su [14] also considered the following iterative scheme in either a uniformly smooth Banach space or a reflexive Banach space having a weakly sequentially continuous duality mapping, which is a simpler modification of the iterative scheme (1.2):

$$\begin{cases} x_0 = x \in E, \\ y_n = \beta_n x_n + (1 - \beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n \end{cases} \tag{1.4}$$

where $u \in \overline{D(A)}$ is an arbitrary (but fixed) element and sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$. They proved that $\{x_n\}$ generated by (1.4) converges strongly to a zero of m-accretive operator A under the conditions (H1), (H2) and (H3) on $\{\alpha_n\}$ and $\{\beta_n\}$, and the condition (H5) on $\{r_n\}$. Very recently, Ceng et al. [15] studied the following composite iterative scheme in the same Banach spaces:

$$\begin{cases} x_0 = x \in E, \\ y_n = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n J_{r_n} y_n, \end{cases} \tag{1.5}$$

where $u \in \overline{D(A)}$ is an arbitrary (but fixed) element, under the control conditions (H1), (H2), (H3), (H5) and

(H6) $\beta_n \in [0, a)$ for some $a \in (0, 1)$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

On the other hand, as the viscosity approximation method [16,17], Chen and Zhu [18,19] considered the iterative scheme: for resolvent J_{r_n} of m-accretive operator A, $f \in \Sigma_C$ $(C = \overline{D(A)})$ and $\alpha_n \in (0, 1)$,

$$\begin{cases} x_0 = x \in C \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0. \end{cases} \tag{1.6}$$