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Monotone iterative method for initial value problem involving Riemann–Liouville fractional derivatives

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ABSTRACT

In this paper, using monotone iterative method, we consider the existence and uniqueness of solution of the initial value problem for fractional differential equation involving Riemann–Liouville fractional derivative.

1. Introduction

We will devote to considering the existence and uniqueness of solution of the following initial value problem for fractional differential equation, using the method of upper and lower solutions and its associated monotone iterative

$$\begin{cases} D^{\alpha}u(t)=f(t,u), & t\in(0,T],\\ t^{1-\alpha}u(t)|_{t=0}=u_0, \end{cases} \tag{1.1}$$

where $0<T<+\infty$, and D^{α} is Riemann–Liouville fractional derivative of order $0<\alpha<1$ defined by (see [1])

$$D^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-s)^{-\alpha}u(s)\mathrm{d}s=\frac{\mathrm{d}}{\mathrm{d}t}I^{1-\alpha}u(t),$$

here

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}u(s)\mathrm{d}s=I^{1-\alpha}u(t)$$

is Riemann–Liouville fractional integral of order $1-\alpha$, see [1].

Differential equations of fractional order occur more frequently in different research areas and engineering, such as physics, chemistry, control of dynamical systems etc. Recently, many people paid attention to existence result of solution of the initial value problem for fractional differential equations, such as [2–12].

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The monotone iterative technique, combined with the method of upper and lower solutions, is a powerful tool for proving the existence of solutions of nonlinear differential equations, such as [13–20]. In [20], the differential equation of (1.1) with initial value condition $u(0) = u_0$ was discussed

$$\begin{cases} D^\alpha u(t) = f(t, u), & t \in (0, T], \\ u(0) = u_0, \end{cases} \tag{1.2}$$

here D^α is Riemann–Liouville fractional derivative of order $0 < \alpha < 1$. In this paper, authors considered the existence of solution and extremal solutions of problem (1.2), employing the classical approach. The idea of this paper is very interesting, but we find out that this paper is wrong in two aspects. One is: for $f \in C([0, T], R)$, the authors say that problem (1.2) is equivalent to the following Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, u(s)) \mathrm{d}s, \quad 0 \le t \le T, \tag{1.3}$$

and that, all the following verification depends on (1.3). However, problem (1.2) is not equivalent to integral equation (1.3) (in fact, problem (1.2) cannot be deduced to an integral equation), because D^α is the Riemann–Liouville fractional derivative of order $0 < \alpha < 1$. From [1], we know that the Riemann–Liouville fractional derivative of order $0 < \alpha < 1$ of constant u_0 is not zero, but is $\frac{u_0 t^{-\alpha}}{\Gamma(1-\alpha)}$, $t > 0$; the other is : initial value condition $u(0) = u_0$ is not suitable (expect $u_0 = 0$), since fractional differential equation (involving the Riemann–Liouville fractional derivative) cannot have such initial value condition, but the initial value condition $t^{1-\alpha} u(t)|_{t=0}$ or $I^{1-\alpha} u(t)|_{t=0}$ ($t^{1-\alpha} u(t)|_{t=0}$ and $I^{1-\alpha} u(t)|_{t=0}$ can transform each other, see [1]), that is, problem (1.2) is not a suitable problem (expect $u_0 = 0$).

Remark 1.1. Initial value problem

$$\begin{cases} D^s u(t) = f(u), & 0 < t \le T, \\ u(a) = u_0, \end{cases}$$

where $0 < a < T$, has meaning.

Remark 1.2 (*[4]*). Let $0 < s < 1$. Assume $f(x) \in C(R^+) \cap L^1_{\mathrm{loc}}(R^+)$. Then for all $(a, b) \in R^+ \times R$, the initial value problem

$$\begin{cases} D^s u(x) = f(x), \\ u(a) = b, \end{cases}$$

has a unique solution in $C(R^+) \cap L^1_{\mathrm{loc}}(R^+)$ given by

$$u(x) = \left(b - \frac{1}{\Gamma(s)} \int_0^a (a - t)^{s-1} f(t) \mathrm{d}t \right) \frac{x^{s-1}}{a^{s-1}} + \frac{1}{\Gamma(s)} \int_0^x (x - t)^{s-1} f(t) \mathrm{d}t.$$

Remark 1.3 (*[1]*). Let $0 < \alpha < 1$ and let $y(x) \in C_{1-\alpha}([0, b])$.

(a) If

$$\lim_{x \to 0^+} [x^{1-\alpha} y(x)] = c, \quad c \in R,$$

then

$$I^{1-\alpha} y(0+) := \lim_{x \to 0^+} I^{1-\alpha} y(x) = c\Gamma(\alpha).$$

(b) If

$$\lim_{x \to 0^+} I^{1-\alpha} y(x) = b, \quad b \in R$$

and if there exists the limit $\lim_{x \to 0^+} [x^{1-\alpha} y(x)]$, then

$$\lim_{x \to 0^+} [x^{1-\alpha} y(x)] = \frac{b}{\Gamma(\alpha)}.$$

In this paper, we will use monotone iterative method to give a result of existence and uniqueness of the solution of problem (1.1). In this result, we delete the monotone demand of f in [20].

Definition 1.1. We call a function $u(t)$ a classical solution of problem (1.1), if:
(i) $u(t)$ is continuous on $(0, T]$, $t^{1-\alpha} u(t)$ is continuous on $[0, T]$, and its fractional integral $I^{1-\alpha} u(t)$ is continuously differentiable for $t \in (0, T]$;
(ii) $u(t)$ satisfies problem (1.1).

Let $C_{1-\alpha}([0, T]) = \{u \in C(0, T]; t^{1-\alpha} u \in C([0, T])\}$.